For Online Publication: Technical Appendix to Accompany

"Revealing Transactions Data to Third Parties: Implications of Privacy Regimes for Welfare in Online Markets"

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This Technical Appendix characterizes the perfect Bayesian equilibria that arise in settings where a consumer interacts sequentially with two merchants. The consumer (or buyer, B) is sophisticated (i.e., fully rational). B interacts first with merchant (or seller) $S1$, and then with merchant $S2$. B's reservation value for each unit of each seller's product is $r \in \{ \underline{r}, \overline{r} \}$.

We first characterize outcomes under privacy, where it is common knowledge that no transactions data from B's interaction with $S1$ will be revealed to $S2$. Then we characterize outcomes under transparency, where it is common knowledge that all transactions data from B's interaction with $S1$ will be revealed to $S2$.

Lemmas 1 – 4 in the text follow directly from Claims 1 – 3 below and the explanations in the text. Lemmas 5 and 6 in the text follow from the remaining theorems and corollaries in this Appendix. Specifically, Lemma 5 follows from Theorems 1 – 4, 9, 10, 14, and 17 and their corollaries. Lemma 6 follows from Theorems 5 – 8, 11 – 13, 15, 16, and 18 and their corollaries.

Claim 1. *Under privacy, Si maximizes her expected profit by setting price $p_i = \overline{r}$ if $c_i > \widehat{c} \equiv \overline{r} - \frac{\overline{r} - \underline{r}}{1-\phi}$ and price $p_i = \underline{r}$ if $c_i \leq \widehat{c}$.*

Proof. Si earns 0 with probability one if she sets $p_i > \overline{r}$. Her (expected) payoff is $[\, \overline{r} - c_i \,] \, \phi \, n_i$ if she sets $p_i = \overline{r}$. Si earns certain payoff $[\, \underline{r} - c_i \,] \, n_i$ if she sets $p_i = \underline{r}$. Because Si's payoff is strictly less than $[\, \underline{r} - c_i \,] \, n_i$ if she sets $p_i < \underline{r}$ and strictly less than $[\, \overline{r} - c_i \,] \, \phi \, n_i$ if she sets any $p_i \in (\, \underline{r}, \overline{r} \,)$, Si's optimal price is either $p_i = \underline{r}$ or $p_i = \overline{r}$.

Comparing the payoffs from these two prices reveals that Si secures a strictly higher expected payoff by setting $p_i = \overline{r}$ if and only if:

$$[\, \overline{r} - c_i \,] \, \phi \, n_i \; > \; [\, \underline{r} - c_i \,] \, n_i$$

$$\Leftrightarrow \quad c_i \; > \; \frac{\underline{r} - \phi \, \overline{r}}{1 - \phi} \; = \; \frac{[\, 1 - \phi \,] \, \overline{r} - [\, \overline{r} - \underline{r} \,]}{1 - \phi} \; = \; \overline{r} - \frac{\overline{r} - \underline{r}}{1 - \phi} \; \equiv \; \widehat{c} \, .$$

Similarly, Si optimally sets $p_i = \underline{r}$ if $c_i \leq \widehat{c}$. ■

Claim 2. *Under privacy, Si's equilibrium payoff is: (i) $\phi \, n_i \, [\, \overline{r} - c_i \,]$ if $c_i > \widehat{c}$; and (ii) $n_i \, [\, \underline{r} - c_i \,]$ if $c_i \leq \widehat{c}$.*

Proof. The conclusion follows directly from Claim 1. ■

Claim 3. *Under privacy, B's equilibrium welfare is:*

$$\begin{array}{lllllll} 0 & \text{if} & r = \underline{r} & & & & \\ 0 & \text{if} & r = \overline{r} & \text{and} & c_1 > \widehat{c} & \text{and} & c_2 > \widehat{c} \\ [\,\overline{r} - \underline{r}\,]\, n_1 & \text{if} & r = \overline{r} & \text{and} & c_1 \leq \widehat{c} & \text{and} & c_2 > \widehat{c} \\ [\,\overline{r} - \underline{r}\,]\, n_2 & \text{if} & r = \overline{r} & \text{and} & c_1 > \widehat{c} & \text{and} & c_2 \leq \widehat{c} \\ [\,\overline{r} - \underline{r}\,]\,[\,n_1 + n_2\,] & \text{if} & r = \overline{r} & \text{and} & c_1 \leq \widehat{c} & \text{and} & c_2 \leq \widehat{c}\,. \end{array} \tag{1}$$

Proof. The conclusion follows directly from Claim 1. ■

To prove Lemmas 5 and 6 in the paper, we now characterize the perfect Bayesian equilibria that can arise under transparency for each of the possible configurations of the sellers' production costs. The following definitions are employed in the ensuing analysis.

Definitions

1. $\phi_{n_1}(p_1)$ is $S2$'s *ex post* assessment of the probability that $r = \overline{r}$ after observing B purchase $n_1 > 0$ units from $S1$ at price p_1.
2. $\phi_0(p_1)$ is $S2$'s *ex post* assessment of the probability that $r = \overline{r}$ after after observing B purchase 0 units from $S1$ at price p_1.
3. $c^* \equiv \widehat{c} + \phi \frac{n_2}{n_1} \left[\frac{\overline{r} - \underline{r}}{1 - \phi} \right]$.
4. A separating equilibrium is one in which B's action in his interaction with $S1$ varies with his reservation value, r.
5. A pooling equilibrium is one in which B's action in his interaction with $S1$ does not vary with his reservation value, r.

We begin by presenting three conclusions that hold for all possible configurations of the sellers' costs.

Theorem 1. *Suppose $n_1 \leq n_2$. Then a separating perfect Bayesian equilibrium does not exist under transparency.*

Proof. Initially suppose a separating equilibrium exists in which B buys from $S1$ at some $\widetilde{p}_1$ if and only if $r = \overline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 1$ and $\phi_0(\widetilde{p}_1) = 0$. Consequently, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$. B will buy from $S2$ if and only if the price she sets does not exceed B's valuation, r.

First suppose $r = \overline{r}$. B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1\,[\,\overline{r} - \widetilde{p}_1\,] + n_2\,[\,\overline{r} - \overline{r}\,] \;=\; n_1\,[\,\overline{r} - \widetilde{p}_1\,]\,. \tag{2}$$

B's welfare if he does not buy from $S1$ at price $p_1 = \widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \overline{r}) \;=\; n_2\,[\,\overline{r} - \underline{r}\,]\,. \tag{3}$$

(2) and (3) imply that B will buy from $S1$ when $p_1 = \widetilde{p}_1$ if and only if:

$$n_1\,[\,\overline{r} - \widetilde{p}_1\,] \;\geq\; n_2\,[\,\overline{r} - \underline{r}\,] \quad\Leftrightarrow\quad n_1\,\overline{r} \;-n_1\,\widetilde{p}_1 \;\geq\; n_2\,[\,\overline{r} - \underline{r}\,]$$

$$\Leftrightarrow\quad n_1\,\overline{r} - \;n_2\,[\,\overline{r} - \underline{r}\,] \;\geq\; n_1\,\widetilde{p}_1 \quad\Leftrightarrow\quad \overline{r} - \frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,] \;\geq\; \widetilde{p}_1\,.$$

Because $n_2 \geq n_1$:

$$\overline{r} - \frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,] \;\leq\; \overline{r} - \frac{n_2}{n_2}\,[\,\overline{r} - \underline{r}\,] \;=\; \underline{r}\,.$$

Therefore, if such a separating equilibrium exists, B buys from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if:

$$\widetilde{p}_1 \;\leq\; \overline{r} - \;\frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,] \;\leq\; \underline{r}\,. \tag{4}$$

Now suppose $r = \underline{r}$. B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \;=\; n_1\,[\,\underline{r} - \widetilde{p}_1\,]\,. \tag{5}$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \underline{r}) \;=\; n_2\,[\,\underline{r} - \underline{r}\,] \;=\; 0\,. \tag{6}$$

(5) and (6) imply that B will not buy from $S1$ at price, $\widetilde{p}_1$ if and only if:

$$n_1\,[\,\underline{r} - \;\widetilde{p}_1\,] \;<\; 0 \quad\Leftrightarrow\quad \widetilde{p}_1 \;>\; \underline{r}\,. \tag{7}$$

The last inequality in (7) contradicts (4). Therefore, when $n_2 \geq n_1$, there does not exist a separating equilibrium in which B buys from $S1$ at some price $\widetilde{p}_1$ if and only if $r = \overline{r}$.

To conclude the proof, suppose a separating equilibrium exists in which B buys from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if $r = \underline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 0$ and $\phi_0(\widetilde{p}_1) = 1$. Consequently, $S2$ will set price $p_2 = \underline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \overline{r}$ if B does not buy from $S1$.

First suppose $r = \overline{r}$. B's welfare if she buys from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1\,[\,\overline{r} - \widetilde{p}_1\,] + n_2\,[\,\overline{r} \;-\; \underline{r}\,]\,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \overline{r}) \;=\; n_2\,[\,\overline{r} - \overline{r}\,] = \;0.$$

Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1\,[\,\overline{r} \;-\widetilde{p}_1\,] + n_2\,[\,\overline{r} \;-\; \underline{r}\,] \;<\; 0 \quad\Leftrightarrow\quad [\,\overline{r} \;-\widetilde{p}_1\,] + \frac{n_2}{n_1}\,[\,\overline{r} \;-\; \underline{r}\,] \;<\; 0$$

$$\Leftrightarrow\quad \overline{r} + \frac{n_2}{n_1}\,[\,\overline{r} \;-\; \underline{r}\,] \;<\; \widetilde{p}_1\,.$$

Because $\overline{r} < \overline{r} \,+\, \frac{n_2}{n_1}\,[\,\overline{r} \;-\; \underline{r}\,]$, B will not buy from from $S1$ at price $\widetilde{p}_1$ if and only if:

$$\overline{r} \; < \; \overline{r} + \frac{n_2}{n_1}\,[\,\overline{r} \,-\, \underline{r}\,] \; < \; \widetilde{p}_1. \tag{8}$$

Now suppose $r = \underline{r}$. B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ (and subsequently buys from $S2$ at price $p_2 = \underline{r}$) is:

$$\pi_B^{n_1}(\,\widetilde{p}_1, \underline{r}\,) \;=\; n_1\,[\,\underline{r} \,-\, \widetilde{p}_1] + n_2\,[\,\underline{r} - \underline{r}\,] \;=\; n_1\,[\,\underline{r} \,-\, \widetilde{p}_1\,]\,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) is 0. Therefore, B will buy from $S1$ at $\widetilde{p}_1$ if and only if:

$$n_1\,[\,\underline{r} \,-\, \widetilde{p}_1\,] \;\geq\; 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \;\leq\; \underline{r}\,. \tag{9}$$

(9) contradicts (8). Therefore, there does not exist a separating equilibrium when $n_2 \geq n_1$. ■

Theorem 2. *There does not exist a perfect Bayesian equilibrium under transparency in which: (i) $S1$ sets $p_1 = \overline{r}$; (ii) B buys from S_1 if and only if $r = \overline{r}$; (iii) $S2$ always sets $p_2 = \overline{r}$.*

Proof. Suppose there exists a perfect Bayesian equilibrium in which $S1$ sets $p_1 = \overline{r}$, B buys at this price if and only if $r = \overline{r}$, and $S2$ always sets $p_2 = \overline{r}$. Suppose $S2$ observes that B did not buy from $S1$ at price $p_1 = \overline{r}$. Because $S2$'s beliefs must satisfy Bayes' Rule along the equilibrium path, $\phi_0(\,\overline{r}\,) = 0$. Consequently, $S2$'s payoff is $n_2\,[\,\underline{r} \,-\, c_2\,] > 0$ if she sets $p_2 = \underline{r}$ and 0 if she sets $p_2 = \overline{r}$. The fact that $S2$'s payoff is strictly higher when she sets $p_2 = \underline{r}$ after seeing that B did not buy from $S1$ at $p_1 = \overline{r}$ contradicts the premise that $S2$ always sets $\overline{r}$ in the putative equilibrium. ■

Theorem 3. *Suppose $n_1 > n_2$. Then under transparency, $S1$ will set $p_1 = \widehat{p}_1 \equiv \overline{r} - \frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,]$ in any separating perfect Bayesian equilibrium.*

Proof. First consider a separating equilibrium in which B buys from $S1$ when she sets price $\widetilde{p}_1$ if and only if $r = \overline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\,\widetilde{p}_1\,) = 1$ and $\phi_0(\,\widetilde{p}_1\,) = 0$. Consequently, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$ and $S2$ will set $p_2 = \underline{r}$ otherwise.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\,\widetilde{p}_1,\, \overline{r}\,) \;=\; n_1\,[\,\overline{r} - \,\widetilde{p}_1] + n_2\,[\,\overline{r} - \,\overline{r}\,] \;=\; n_1\,[\,\overline{r} - \,\widetilde{p}_1]\,.$$

B's welfare if he does not buy from $S1$ at price $p_1 = \widetilde{p}_1$ is:

$$\pi_B^{0}(\,\widetilde{p}_1,\, \overline{r}\,) \;=\; n_2\,[\,\overline{r} - \underline{r}\,]\,.$$

Therefore, B will buy from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if:

$$n_1\,[\,\overline{r} - \,\widetilde{p}_1] \;\geq\; n_2\,[\,\overline{r} - \underline{r}\,] \quad \Leftrightarrow \quad \widetilde{p}_1 \;\leq\; \overline{r} - \frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,] \equiv \widehat{p}_1\,. \tag{10}$$

B's welfare if he buys from from $S1$ at price $p_1 = \widetilde{p}_1$ when $r = \underline{r}$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \ = \ n_1 [\, \underline{r} - \widetilde{p}_1 \,] \, .$$

B's welfare if he does not buy from $S1$ at price $p_1 = \widetilde{p}_1$ is:

$$\pi_B^{0}(\widetilde{p}_1, \underline{r}) \ = \ n_2 [\, \underline{r} - \underline{r} \,] \ = \ 0 \, .$$

Therefore, B will not buy from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if:

$$n_1 [\, \underline{r} - \widetilde{p}_1] \ < \ 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \ < \ \underline{r} \, . \tag{11}$$

(10) and (11) imply that $S1$'s price must lie in the interval $(\underline{r}\,, \widehat{p}_1]$ in any separating equilibrium in which B buys from $S1$ at price $\widetilde{p}_1 \in (\underline{r}\,, \widehat{p}_1]$ if and only if $r = \overline{r}$. (10) and (11) also imply that when $S1$ sets $\widetilde{p}_1 \in (\underline{r}, \, \widehat{p}_1)$, her payoff is $\phi \, n_1 [\, \widetilde{p}_1 - c_1] \ < \ \phi \, n_1 [\, \widehat{p}_1 - c_1 \,]$. Because $S1$ can secure a higher payoff by charging $\widehat{p}_1$ than by charging $\widetilde{p}_1 \in (\underline{r}\,, \widehat{p}_1)$, any such $\widetilde{p}_1$ cannot arise in a separating equilibrium. Consequently, $\widetilde{p}_1 = \widehat{p}_1$ in any separating equilibrium in which B buys from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if $r = \overline{r}$.

Now suppose a separating equilibrium exists in which B buys from $S1$ at price $p_1 = \widetilde{p}_1$ if and only if $r = \underline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 0$ and $\phi_0(\widetilde{p}_1) = 1$. Consequently, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \ \overline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \ = \ n_1 [\, \overline{r} - \widetilde{p}_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \, .$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \ = \ n_2 [\, \overline{r} - \overline{r} \,] = \ 0.$$

Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\, \overline{r} - \widetilde{p}_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \ < \ 0 \ \Leftrightarrow \quad \widetilde{p}_1 \ > \ \overline{r} + \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \ > \ \overline{r}. \tag{12}$$

B's welfare if he buys from $S1$ at price p_1 when $r = \underline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \, \underline{r}) \ = \ n_1 [\, \underline{r} - \widetilde{p}_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] \ = \ n_1 [\, \underline{r} - \widetilde{p}_1 \,] \, .$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) is 0. Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\, \underline{r} - \widetilde{p}_1] \ \geq \ 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \ \leq \ \underline{r} \, . \tag{13}$$

(13) provides a contradiction of (12). Therefore, no separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \underline{r}$. ■

We now characterize the equilibria that can arise for each of the possible configurations of the sellers' costs (and relative values of n_1 and n_2).

<u>**Setting 1A**</u>. $c_1 > \widehat{c}$, $c_2 \le \widehat{c}$, and $n_1 > n_2$.

Theorem 4. *Suppose $n_1 > n_2$, $c_1 > c^* > \widehat{c}$, and $c_2 \le \widehat{c}$. Then under transparency, a separating perfect Bayesian equilibrium exists in which: (i) $S1$ sets $\widehat{p}_1 \equiv \overline{r} - \frac{n_2}{n_1}[\overline{r} - \underline{r}]$; (ii) B buys from $S1$ if and only if $r = \overline{r}$; (iii) $S2$ sets $p_2 = \overline{r}$ if B buys from $S1$, and sets $p_2 = \underline{r}$ if B does not buy from $S1$; and (iv) B buys from $S2$ if and only $p_2 \le r$.*

<u>Proof</u>. The beliefs that support the identified equilibrium actions are: (i) $\phi_{n_1}(\underline{r}) = \phi$; (ii) $\phi_0(\underline{r}) \le \phi$; (iii) $\phi_{n_1}(p_1) = 1$ and $\phi_0(p_1) \le \phi$ for all $p_1 > \overline{r}$; (iv) $\phi_{n_1}(p_1) = 1$ and $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}]$; and (v) $\phi_{n_1}(p_1) = \phi$ and $\phi_0(p_1) \le \phi$ for all $p_1 < \underline{r}$.

The proof proceeds by backward induction. We first prove that $S2$'s equilibrium actions are optimal, given her beliefs. Then we prove that B's equilibrium actions are optimal, given $S2$'s beliefs. Next we prove that $S1$'s equilibrium actions are optimal. Finally, we verify that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

A. <u>Prove that $S2$'s equilibrium actions are optimal</u>.

$\phi_{n_1}(\widehat{p}_1) = 1$. Therefore, $S2$ maximizes her payoff by setting $p_2 = \overline{r}$ if B buys from $S1$ at price $p_1 = \widehat{p}_1$.

$\phi_0(\widehat{p}_1) = 0$. Therefore, $S2$ optimally sets $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 = \widehat{p}_1$.

B. <u>Prove that B's equilibrium strategy is optimal</u>.

Because the game ends following B's interaction with $S2$, B maximizes his welfare by buying from $S2$ if and only if $p_2 \le r$.

We now prove that B maximizes his welfare by buying from $S1$ at price $p_1 = \widehat{p}_1$ if and only if $r = \overline{r}$.

First suppose $r = \overline{r}$. If B buys from $S1$ at price $p_1 = \widehat{p}_1$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\widehat{p}_1) = 1$. Therefore, B's welfare is:

$$\begin{aligned} \pi_B^{n_1}(\widehat{p}_1, \overline{r}) &= n_1[\overline{r} - \widehat{p}_1] + n_2[\overline{r} - \overline{r}] \;=\; n_1[\overline{r} - \widehat{p}_1] \\ &= n_1\left[\overline{r} - \left(\overline{r} - \frac{n_2}{n_1}[\overline{r} - \underline{r}]\right)\right] \;=\; n_2[\overline{r} - \underline{r}]. \end{aligned}$$

If B does not buy from $S1$ when $p_1 = \widehat{p}_1$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\widehat{p}_1) = 0$. In this case, B's welfare is:

$$\pi_B^0(\widehat{p}_1, \overline{r}) \;=\; n_2[\overline{r} - \underline{r}].$$

Because $\pi_B^{n_1}(\widehat{p}_1, \overline{r}) = \pi_B^0(\widehat{p}_1, \overline{r})$, B optimally buys from $S1$ when she sets $p_1 = \widehat{p}_1$ when $r = \overline{r}$.

Now suppose $r = \underline{r}$. If B buys from $S1$ at price $p_1 = \widehat{p}_1$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\widehat{p}_1) = 1$. B will not buy from $S2$ because $\underline{r} < p_2$. Therefore, B's welfare is:

$$\pi_B^{n_1}(\widehat{p}_1, \underline{r}) \;=\; n_1 \left[\underline{r} - \widehat{p}_1 \right] \;<\; 0 \,.$$

If B instead does not buy from $S1$ at price $p_1 = \widehat{p}_1$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\widehat{p}_1) = 0$. Therefore, B's welfare is:

$$\pi_B^{0}(\widehat{p}_1, \underline{r}) \;=\; n_2 \left[\underline{r} - \underline{r} \right] \;=\; 0.$$

Because $\pi_B^0(\widehat{p}_1, \underline{r}) > \pi_B^{n_1}(\widehat{p}_1, \underline{r})$, B optimally does not buy from $S1$ when $p_1 = \widehat{p}_1$ and $r = \underline{r}$.

C. Prove that $S1$'s equilibrium actions are optimal.

1. We begin by characterizing B's optimal response to out-of-equilibrium prices by $S1$.

Result C1. When $r = \overline{r}$, B optimally does not buy from $S1$ at any price $p_1 > \widehat{p}_1$, and buys from $S1$ at any price $p_1 < \widehat{p}_1$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \overline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 \left[\overline{r} - p_1 \right] + n_2 \left[\overline{r} - \overline{r} \right] \;=\; n_1 \left[\overline{r} - p_1 \right] \;<\; 0 \,.$$

Because $\phi_0(p_1) \le \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \le \widehat{c}$, by assumption). Therefore, B's welfare is:

$$\pi_B^{0}(p_1, \overline{r}) \;=\; n_2 \left[\overline{r} - \underline{r} \right] \;>\; 0 \,.$$

Consequently, when $r = \overline{r}$, B optimally does not buy from $S1$ if $p_1 > \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\widehat{p}_1, \overline{r}]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 \in (\widehat{p}_1, \overline{r}]$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 \left[\overline{r} - p_1 \right] + n_2 \left[\overline{r} - \overline{r} \right] \;<\; n_1 \left[\overline{r} - \widehat{p}_1 \right] \;=\; n_2 \left[\overline{r} - \underline{r} \right].$$

Because $\phi_0(p_1) = 0$ for all $p_1 \in (\widehat{p}_1, \overline{r}]$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$. Consequently, B's welfare is:

$$\pi_B^{0}(p_1, \overline{r}) \;=\; n_2 \left[\overline{r} - \underline{r} \right].$$

Because $\pi_B^{n_1}(p_1, \overline{r}) < \pi_B^0(p_1, \overline{r})$, B will not buy from $S1$ at any price $p_1 \in (\widehat{p}_1, \overline{r}]$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1)$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 \in (\underline{r}, \widehat{p}_1)$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 \left[\overline{r} - p_1 \right] + n_2 \left[\overline{r} - \overline{r} \right] \;>\; n_1 \left[\overline{r} - \widehat{p}_1 \right] \;=\; n_2 \left[\overline{r} - \underline{r} \right].$$

Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \widehat{p}_1)$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$. Consequently, B's welfare is:

$$\pi_B^{0}(p_1, \overline{r}) \;=\; n_2 \left[\overline{r} - \underline{r} \right].$$

Because $\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r})$, B will buy from $S1$ at any price $p_1 \in (\underline{r}, \widehat{p}_1)$ when $r = \overline{r}$.

Finally, suppose that $S1$ sets $p_1 \le \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \le \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ (since $c_2 \le \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \;\geq\; [\, n_1 + n_2 \,] [\, \overline{r} - \underline{r} \,] \,.$$

Because $\phi_0(p_1) \leq \phi$ for all $p_1 \leq \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ (since $c_2 \leq \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}\,) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \,.$$

Because $\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r}\,)$, B will buy from $S1$ at any price $p_1 \leq \underline{r}$ when $r = \overline{r}$

Result C2. When $r = \underline{r}$ and $p_1 \neq \widehat{p}$, B optimally does not buy from $S1$ at any price $p_1 > \underline{r}$ and buys from $S1$ at any price $p_1 \leq \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \underline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$, $S2$ will set $p_2 = \overline{r}$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \underline{r}) \;=\; n_1 [\, \underline{r} - \; p_1 \,] + 0\, [\, \overline{r} - \underline{r} \,] \;=\; n_1 [\, \underline{r} - p_1 \,] \;<\; 0 \,.$$

Because $\phi_0(p_1) \leq \phi$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ (since $c_2 < \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^0(p_1, \underline{r}\,) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;=\; 0 \,.$$

Because $\pi_B^{n_1}(p_1, \underline{r}) < \pi_B^0(p_1, \underline{r}\,)$, B will not buy from $S1$ at any price $p_1 > \underline{r}$ when $r = \underline{r}$.

Next suppose that $S1$ sets $p_1 \leq \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$, if B buys from $S1$, $S2$ will set $p_2 = \underline{r}$ (since $c_2 < \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \underline{r}) \;=\; n_1 [\, \underline{r} - \; p_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] \;=\; n_1 [\, \underline{r} - p_1 \,] \;\geq\; 0 \,.$$

Because $\phi_0(p_1) \leq \phi$ for all $p_1 \leq \underline{r}$, $S2$ will set $p_2 = \underline{r}$ (since $c_2 < \widehat{c}$) if B does not buy from $S1$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \underline{r}\,) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;=\; 0 \,.$$

Because $\pi_B^{n_1}(p_1, \underline{r}) \geq \pi_B^0(p_1, \underline{r}\,)$, B will buy from $S1$ at any price $p_1 \leq \underline{r}$ when $r = \underline{r}\,$.

2. We now prove that $S1$'s equilibrium actions are optimal.

When $S1$ sets $p_1 = \widehat{p}_1$, B will buy from $S1$ if and only if $r = \overline{r}$. Consequently, $S1$'s payoff is:

$$\pi_1(\widehat{p}_1) \;=\; \phi\, n_1 [\, \widehat{p}_1 - c_1] \;>\; 0 \,.$$

Results C1 and C2 imply that B will not buy from $S1$ at any price $p_1 > \widehat{p}_1$. Consequently, $S1$'s payoff is $\pi_1(p_1) = 0$ for all $p_1 > \widehat{p}_1$.

Results C1 and C2 imply that if $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1)$, B will buy from $S1$ if $r = \overline{r}$ and not buy from $S1$ if $r = \underline{r}$. Consequently, $S1$'s payoff is:

$$\pi_1(p_1) \;=\; \phi\, n_1 [\, p_1 - c_1 \,] \;<\; \phi\, n_1 [\, \widehat{p}_1 - c_1 \,] \,.$$

Results C1 and C2 imply that B will buy from $S1$ if she sets $p_1 \leq \underline{r}$. Consequently, $S1$'s payoff from a price $p_1 \leq \underline{r}$ is:

$$\pi_1(p_1) \;=\; n_1 [\, p_1 - c_1 \,] \;\leq\; n_1 [\, \underline{r} - c_1] \;<\; \phi\, n_1 [\, \widehat{p}_1 - c_1 \,] \,. \tag{14}$$

The inequality in (14) holds because:

$$\underline{r} - c_1 \;<\; \phi\,[\,\widehat{p}_1 - c_1\,] \quad\Leftrightarrow\quad [\,1-\phi\,]\,c_1 \;>\; \underline{r} - \phi\,\widehat{p}_1 \;=\; \underline{r} - \phi\left[\overline{r} - \frac{n_2}{n_1}\,(\,\overline{r} - \underline{r}\,)\right]$$

$$\Leftrightarrow\quad [\,1-\phi\,]\,c_1 \;>\; \underline{r} - \phi\,\overline{r} + \phi\,\frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,]$$

$$\Leftrightarrow\quad [\,1-\phi\,]\,c_1 \;>\; [\,1-\phi\,]\,\overline{r} - (\,\overline{r} - \underline{r}\,) + \phi\,\frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,]$$

$$\Leftrightarrow\quad c_1 \;>\; \overline{r} - \frac{\overline{r} - \underline{r}}{1-\phi} + \phi\,\frac{n_2}{n_1}\,[\,\overline{r} - \underline{r}\,] \;=\; \widehat{c} + \phi\,\frac{n_2}{n_1}\left[\frac{\overline{r} - \underline{r}}{1-\phi}\right]. \tag{15}$$

The inequality in (15) holds because $c_1 > c^*$ by hypothesis. Therefore, $S1$ maximizes her payoff by setting $p_1 = \widehat{p}_1$.

D. Prove that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

$$\Pr(r = \overline{r} \mid B \text{ buys at } \widehat{p}_1) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ buys at } \widehat{p}_1)}{\Pr(\text{Buys at } \widehat{p}_1)}$$

$$= \frac{\Pr(B \text{ buys at } \widehat{p}_1 \mid r = \overline{r}\,)\,\Pr(r = \overline{r})}{\Pr(B \text{ buys at } \widehat{p}_1 \mid r = \overline{r}\,)\,\Pr(r = \overline{r}) + \Pr(\,B \text{ buys at } \widehat{p}_1 \mid r = \underline{r}\,)\,\Pr(r = \underline{r}\,)}$$

$$= \frac{1\,[\,\phi\,]}{1\,[\,\phi\,] + 0\,[\,1-\phi\,]} \;=\; 1 \;=\; \phi_{n_1}(\,\widehat{p}_1).$$

$$\Pr(r = \overline{r} \mid B \text{ does not buy at } \widehat{p}_1) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ does not buy at } \widehat{p}_1)}{\Pr(\,B \text{ does not buy at } \widehat{p}_1)}$$

$$= \frac{\Pr(B \text{ does not buy at } \widehat{p}_1 \mid r = \overline{r}\,)\,\Pr(r = \overline{r})}{\Pr(B \text{ does not buy at } \widehat{p}_1 \mid r = \overline{r}\,)\,\Pr(r = \overline{r}) + \Pr(\,B \text{ does not buy at } \widehat{p}_1 \mid r = \underline{r}\,)\,\Pr(r = \underline{r}\,)}$$

$$= \frac{0\,[\,\phi\,]}{0\,[\,\phi\,] + 1\,[\,1-\phi\,]} \;=\; 0 \;=\; \phi_0(\,\widehat{b}_1). \quad \blacksquare$$

Observation. Theorem 3 implies that the equilibrium identified in Theorem 4 is the unique separating equilibrium under the specified conditions.

Corollary 1. *Under the conditions specified in Theorem 4, B's equilibrium welfare is the same under transparency and privacy for each realization of r.*

Proof. Claim 3 implies that under privacy, B's equilibrium welfare is: (i) 0 when $r = \underline{r}$; and (ii) $n_2\,[\,\overline{r} - \underline{r}\,]$ when $r = \overline{r}$.

Theorem 4 implies that under transparency, B's equilibrium welfare is: (i) $n_1\,[\,\overline{r} - \widehat{p}_1] = n_2\,[\,\overline{r} - \underline{r}\,]$ when $r = \overline{r}$; and (ii) 0 when $r = \underline{r}$. $\blacksquare$

Corollary 2. *Under the conditions specified in Theorem 4, transparency reduces the equilibrium payoff of* $S1$ *and increases the equilibrium payoff of* $S2$ *(by the same amount).*

Proof. Claim 2 implies that under privacy, the payoffs of $S1$ and $S2$ are:

$$\pi_1^V = \phi\, n_1 [\,\overline{r} - c_1] \quad \text{and} \quad \pi_2^V = n_2 [\,\underline{r} - c_2]\,. \tag{16}$$

Theorem 4 implies that under transparency, the payoffs of $S1$ and $S2$, are:

$$\begin{aligned} \pi_1^T &= \phi\, n_1 [\,\widehat{p}_1 - c_1] < \phi\, n_1 [\,\overline{r} - c_1\,], \quad \text{and} \\ \pi_2^T &= \phi\, n_2 [\,\overline{r} - c_2] + [\,1 - \phi\,]\, n_2 [\,\underline{r} - c_2\,] > n_2 [\,\underline{r} - c_2\,]\,. \end{aligned} \tag{17}$$

(16) and (17) imply:

$$\pi_1^T - \pi_1^V = \phi\, n_1 [\,\widehat{p}_1 - \overline{r}\,] = -\,\phi\, n_2 [\,\overline{r} - \underline{r}] = -\left[\,\pi_2^T - \pi_2^V\,\right]. \quad \blacksquare$$

Corollary 3. *Under the conditions specified in Theorem 4, transparency does not affect expected industry welfare.*

Proof. Because industry welfare is the sum of B's welfare and the payoffs of $S1$ and $S2$, the conclusion follows directly from Corollaries 1 and 2. $\blacksquare$

Theorem 5. *Suppose* $c_1 > c^* > \widehat{c}$ *and* $c_2 \leq \widehat{c}$*. Then a pooling perfect Bayesian equilibrium does not exist under transparency.*

Proof. Because $c_2 \leq \widehat{c}$ and $S2$'s beliefs must satisfy Bayes Rule along the equilibrium path, $S2$ will always set $p_2 = \underline{r}$ in any pooling equilibrium. B will always buy from $S2$ at this price. Consequently, B's welfare if he always buys from $S1$ at price p_1 is:

$$\begin{aligned} & n_1 [\,\underline{r} - p_1] \text{ when } r = \underline{r}, \quad \text{and} \\ & n_1 [\,\overline{r} - p_1\,] + n_2 [\,\overline{r} - \underline{r}\,] \text{ when } r = \overline{r}\,. \end{aligned} \tag{18}$$

First consider a pooling equilibrium in which B always buys from $S1$ when $p_1 > \underline{r}$. (18) implies that B's welfare will be negative when $r = \underline{r}$. B can secure nonnegative welfare by not buying from $S1$ at price $p_1 > \underline{r}$ when $r = \underline{r}$. Therefore, a pooling equilibrium in which B always buys from $S1$ at price $p_1 > \underline{r}$ does not exist.

Now consider a pooling equilibrium in which B always buys from $S1$ when $p_1 \leq \underline{r}$. $S1$'s payoff is $n_1 [\,p_1 - c_1\,]$, which is maximized at price $p_1 = \underline{r}$. Consequently, in any such pooling equilibrium, $S1$ will set $p_1 = \underline{r}$ and secure payoff $n_1 [\,\underline{r} - c_1\,] > 0$ (which exceeds $S1$'s payoff in any pooling equilibrium in which B does not buy from $S1$).

(18) implies that when $S1$ sets $p_1 = \underline{r}$, B's welfare is 0 when $r = \underline{r}$ whether he buys or does not buy from $S1$. When $r = \overline{r}$, B's welfare is $[\,n_1 + n_2\,][\,\overline{r} - \underline{r}\,]$ if he buys from $S1$ at price $p_1 = \underline{r}$. His corresponding welfare is $n_2 [\,\overline{r} - \underline{r}\,]$ if he does not buy from $S1$ at price $p_1 = \underline{r}$. Therefore, B will always buy from $S1$ at price $p_1 = \underline{r}$ and $S1$ secures payoff $n_1 [\,\underline{r} - c_1\,]$.

Theorem 4 implies that if $S1$ deviates by setting price $p_1 = \widehat{p}_1$, B will buy at this price when $r = \underline{r}$. Therefore, $S1$'s payoff from this deviation is

$$\pi_1(\widehat{p}_1) \;=\; \phi\, n_1 \left[\, \widehat{p}_1 - c_1 \,\right] \;>\; n_1 \left[\, \underline{r} - c_1 \,\right]$$

where the strict inequality holds because:

$$\underline{r} - c_1 \;<\; \phi \left[\, \widehat{p}_1 - c_1 \,\right] \quad \Leftrightarrow \quad [\,1 - \phi\,]\, c_1 \;>\; \underline{r} - \phi\, \widehat{p}_1 \;=\; \underline{r} - \phi \left[\overline{r} - \frac{n_2}{n_1} \left(\overline{r} - \underline{r} \right) \right]$$

$$\Leftrightarrow \quad [\,1 - \phi\,]\, c_1 \;>\; \underline{r} - \phi\, \overline{r} + \phi\, \frac{n_2}{n_1} \left[\overline{r} - \underline{r} \right]$$

$$\Leftrightarrow \quad [\,1 - \phi\,]\, c_1 \;>\; [\,1 - \phi\,]\, \overline{r} - (\, \overline{r} - \underline{r}\,) + \phi\, \frac{n_2}{n_1} \left[\overline{r} - \underline{r} \right]$$

$$\Leftrightarrow \; c_1 \;>\; \overline{r} - \frac{\overline{r} - \underline{r}}{1 - \phi} + \phi\, \frac{n_2}{n_1} \left[\overline{r} - \underline{r} \right] \;=\; \widehat{c} + \phi\, \frac{n_2}{n_1} \left[\frac{\overline{r} - \underline{r}}{1 - \phi} \right] \;\equiv\; c^* \,.$$

Because S_1 earns a higher payoff by deviating to charge $\widehat{p}_1$, there does not exist a pooling equilibrium under the conditions specified in Theorem 5. ■

Theorem 6. *Suppose $c_1 \in (\widehat{c}, c^*)$ and $c_2 \leq \widehat{c}$. Then the unique perfect Bayesian equilibrium under transparency is the pooling equilibrium in which $S1$ and $S2$ both charge $\underline{r}$ and B always purchases from both sellers at this price.*

Proof. Claims 1 and 2 imply that if a pooling equilibrium exists, $S1$ will set $p_1 = \underline{r}$ and secure payoff $n_1 [\, \underline{r} - c_1 \,]$. Theorems 3 and 4 imply that if a separating equilibrium exists, $S1$ will charge price $\widehat{p}_1$ and secure payoff $\phi\, n_1 [\, \widehat{p}_1 - c_1 \,]$, which is strictly less that $n_1 [\, \underline{r} - c_1 \,]$ when $c_1 \in (\widehat{c}, c^*)$ as demonstrated above. Therefore, $S1$ will set $p_1 = \underline{r}$, B will always buy from $S1$ at this price, and consequently, $S2$ will set $p_2 = \underline{r}$ and B will buy from $S2$ at this price. ■

Observation. Theorems 3, 4, 5, and 6 imply that when $c_2 \leq \widehat{c} < c_1$, the unique perfect Bayesian equilibrium under transparency is: (i) the separating equilibrium in which $S1$ sets $p_1 = \widehat{p}_1$ if $c_1 > c^*$ and (ii) the pooling equilibrium in which $p_1 = p_2 = \underline{r}$ if $c_1 \in (\widehat{c}, c^*)$.

Corollary 4. *Under the conditions specified in Theorem 6, transparency increases B's welfare when $r = \overline{r}$ and does not change B's welfare when $r = \underline{r}$.*

Proof. Claim 3 implies that under privacy, B's equilibrium welfare is: (i) $n_2 [\, \overline{r} - \underline{r} \,]$ when $r = \overline{r}$; and (ii) 0 when $r = \underline{r}$.

Theorem 6 implies that under transparency, B's equilibrium welfare is: (i) $[\, n_1 + n_2 \,] [\, \overline{r} - \underline{r} \,]$ when $r = \overline{r}$; and (ii) 0 when $r = \underline{r}$. ■

Corollary 5. *Under the conditions specified in Theorem 6, transparency: (i) reduces $S1$'s equilibrium payoff; and (ii) does not change $S2$'s equilibrium payoff.*

Proof. Claim 3 implies that under privacy, the payoffs of $S1$ and $S2$ are:

$$\pi_1^V = \phi n_1 [\overline{r} - c_1] \text{ and } \pi_2^V = n_2 [\underline{r} - c_2]. \tag{19}$$

Theorem 6 implies that under transparency, the payoffs of $S1$ and $S2$ are:

$$\pi_i^T = n_i [\underline{r} - c_i] \text{ for } i = 1, 2. \tag{20}$$

The conclusion follows because $\pi_2^V = \pi_2^T$ and $n_1 [\underline{r} - c_1] < \phi n_1 [\overline{r} - c_1]$ since $c_1 > \widehat{c}$. ■

Corollary 6. *Under the conditions specified in Theorem 6, transparency increases equilibrium industry welfare.*

Proof. Corollaries 4 and 5 and their proofs imply that under privacy, equilibrium industry welfare is:

$$W^V = \phi n_2 [\overline{r} - \underline{r}] + \phi n_1 [\overline{r} - c_1] + n_2 [\underline{r} - c_2]. \tag{21}$$

(20) and the proof of Corollary 4 imply that under transparency, equilibrium industry welfare is:

$$W^T = \phi [n_1 + n_2][\overline{r} - \underline{r}] + n_1 [\underline{r} - c_1] + n_2 [\underline{r} - c_2]. \tag{22}$$

(21) and (22) imply:

$$\begin{aligned}
W^T - W^V &= \phi [n_1 + n_2][\overline{r} - \underline{r}] + n_1 [\underline{r} - c_1] - \phi n_2 [\overline{r} - \underline{r}] - \phi n_1 [\overline{r} - c_1] \\
&= \phi [n_1 + n_2][\overline{r} - \underline{r}] - \phi n_2 [\overline{r} - \underline{r}] + n_1 [\underline{r} - c_1] - \phi n_1 [\overline{r} - c_1] \\
&= \phi n_1 [\overline{r} - \underline{r}] - \phi n_1 [\overline{r} - c_1] + n_1 [\underline{r} - c_1] \\
&= \phi n_1 [\overline{r} - \underline{r} - \overline{r} + c_1] + n_1 [\underline{r} - c_1] \\
&= - \phi n_1 [\underline{r} - c_1] + n_1 [\underline{r} - c_1] \\
&= [1 - \phi] n_1 [\underline{r} - c_1] > 0. \quad ■
\end{aligned}$$

Setting 1B. $c_1 > \widehat{c}$, $c_2 \leq \widehat{c}$, and $n_2 \geq n_1$.

Theorem 7. *Suppose $n_2 \geq n_1$, $c_1 > \widehat{c}$, and $c_2 \leq \widehat{c}$. Then under transparency, a pooling perfect Bayesian equilibrium exists in which: (i) $S1$ sets $p_1 = \underline{r}$; (ii) $S2$ sets $p_1 = \underline{r}$; and (iii) B always buys from $S1$ and from $S2$.*

Proof. The beliefs that support the identified equilibrium actions are: (i) $\phi_{n_1}(\underline{r}) = \phi$; (ii) $\phi_0(\underline{r}) \leq \phi$; (iii) $\phi_{n_1}(p_1) = 1$ and $\phi_0(p_1) \leq \phi$ for all $p_1 > \overline{r}$; (iv) $\phi_{n_1}(p_1) = 1$ and $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}]$; and (v) $\phi_{n_1}(p_1) = \phi$ and $\phi_0(p_1) \leq \phi$ for all $p_1 < \underline{r}$.

The proof proceeds by backward induction. We first prove that $S2$'s equilibrium action is optimal, given her beliefs. Then we prove that B's equilibrium actions are optimal, given

$S2$'s beliefs. Next we prove that $S1$'s equilibrium action is optimal. Finally, we verify that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

A. Prove that $S2$'s equilibrium action is optimal.

$\phi_{n_1}(\underline{r}) = \phi$. Therefore, after B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ maximizes her payoff by acting as she does under privacy. Claim 1 implies that because $c_2 \leq \widehat{c}$, $S2$ will set $p_2 = \underline{r}$.

$\phi_0(\underline{r}) \leq \phi$. Therefore, because $c_2 \leq \widehat{c}$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$.

B. Prove that B's equilibrium actions are optimal.

Because the game ends following B's interaction with $S2$, B maximizes his welfare by buying from $S2$ at price $p_2 = \underline{r}$.

We now prove that B maximizes his welfare by buying from $S1$ when she sets price $p_1 = \underline{r}$. First suppose $r = \overline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 \leq \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{n_1}(p_1 = \underline{r}, \overline{r}) \;=\; n_1 [\, \overline{r} - \underline{r}\,] + n_2 [\, \overline{r} - \underline{r}\,] \;=\; [\, n_1 + n_2\,] [\, \overline{r} - \underline{r}\,] .$$

If B instead does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\underline{r}) \leq \phi$ and $c_2 \leq \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{0}(p_1 = \underline{r}, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r}\,] \;<\; [\, n_1 + n_2\,] [\, \overline{r} - \underline{r}\,] .$$

Because $\pi_B^{n_1}(p_1 = \underline{r}, \overline{r}) > \pi_B^{0}(p_1 = \underline{r}, \overline{r})$, B optimally buys from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

Now suppose $r = \underline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 \leq \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{n_1}(p_1 = \underline{r}, \underline{r}) \;=\; n_1 [\, \underline{r} - \underline{r}\,] + n_2 [\, \underline{r} - \underline{r}\,] \;=\; 0 .$$

If B does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_{n_1}(\underline{r}) \leq \phi$ and $c_2 \leq \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{0}(p_1 = \underline{r}, \underline{r}) \;=\; n_2 [\, \underline{r} - \underline{r}\,] \;=\; 0 .$$

Because $\pi_B^{n_1}(p_1 = \underline{r}, \underline{r}) = \pi_B^{0}(p_1 = \underline{r}, \underline{r})$, B optimally buys from $S1$ at price $p_1 = \underline{r}$ when $r = \underline{r}$.

C. Prove that $S1$'s equilibrium action is optimal.

1. We begin by characterizing B's optimal response to out-of-equilibrium prices by $S1$.

Result Ci. When $r = \overline{r}$, B optimally does not buy from $S1$ at any price $p_1 > \underline{r}$ and buys from $S1$ at any price $p_1 < \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \overline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1\,] + n_2 [\, \overline{r} - \overline{r}\,] \;=\; n_1 [\, \overline{r} - p_1\,] \;<\; 0 .$$

Because $\phi_0(p_1) \leq \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \leq \widehat{c}$, by assumption). Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}\,) = \; n_2 \left[\, \overline{r} - \underline{r} \,\right] \; > \; 0 \;\; .$$

Because $\pi_B^0(p_1, \overline{r}\,) > \pi_B^{n_1}(p_1, \overline{r}\,)$, B optimally does not buy from $S1$ when she sets $p_1 > \overline{r}$ and $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\,\underline{r}, \overline{r}\,]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 \in (\,\underline{r}, \overline{r}\,]$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}\,) \; = \; n_1 \left[\, \overline{r} - p_1 \,\right] + n_2 \left[\, \overline{r} - \overline{r} \,\right] \; = \; n_1 \left[\, \overline{r} - p_1 \,\right] \; < \; n_1 \left[\, \overline{r} - \underline{r} \,\right].$$

Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}]$, if B does not buy from $S1$, $S2$ will set $p_2 = \underline{r}$. Because $n_2 \geq n_1$, B's welfare is:

$$\pi_B^0(p_1, \overline{r}\,) \; = \; n_2 \left[\, \overline{r} - \underline{r} \,\right] \; \geq \; n_1 \left[\, \overline{r} - \underline{r} \,\right]. \tag{23}$$

Because $\pi_B^0(p_1, \overline{r}\,) \geq \pi_B^{n_1}(p_1, \overline{r}\,)$, B optimally does not buy from $S1$ when she sets any price $p_1 \in (\,\underline{r}, \overline{r}\,]$ and $r = \overline{r}$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at this price (since $c_2 \leq \widehat{c}$). In this case B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}\,) \; = \; n_1 \left[\, \overline{r} - p_1 \,\right] + n_2 \left[\, \overline{r} - \underline{r} \,\right] \; > \; \left[\, n_1 + n_2 \,\right] \left[\, \overline{r} - \underline{r} \,\right].$$

Because $c_2 \leq \widehat{c}$ and $\phi_0(p_1) \leq \phi$ for all $p_1 < \underline{r}$, if B does not buy from $S1$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}\,) \; = \; n_2 \left[\, \overline{r} - \underline{r} \,\right] \; < \; \left[\, n_1 + n_2 \,\right] \left[\, \overline{r} - \underline{r} \,\right].$$

Because $\pi_B^{n_1}(p_1, \overline{r}\,) > \pi_B^0(p_1, \overline{r}\,)$, B optimally buys from $S1$ for any $p_1 < \underline{r}$ when $r = \overline{r}$.

Result Cii. When $r = \underline{r}$, B optimally does not buy from $S1$ if $p_1 > \underline{r}$ and buys from $S1$ if $p_1 < \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \underline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$, he will subsequently not buy from $S2$ at price $p_2 = \overline{r}$. Consequently, his welfare is $\pi_B^{n_1}(p_1, \underline{r}\,) = n_1 \left[\, \underline{r} - p_1 \,\right] < 0$. Because $\phi_0(p_1) \leq \phi$ for all $p_1 > \underline{r}$, if B does not buy from $S1$ at price $p_1 > \underline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \leq \widehat{c}$). Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 \left[\, \underline{r} - \underline{r} \,\right] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B optimally does not buy from $S1$ when she sets any price $p_1 > \underline{r}$ and $r = \underline{r}$.

Next suppose that $S1$ sets $p_1 \in (\,\underline{r}, \overline{r}\,]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 \in (\,\underline{r}, \overline{r}\,]$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ and he will subsequently not buy from $S2$. Consequently, his welfare is $\pi_B^{n_1}(p_1, \underline{r}\,) = n_1 \left[\, \underline{r} - p_1 \,\right] < 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \in (\,\underline{r}, \overline{r}\,]$, if B does not buy from $S1$ at price $p_1 \in (\,\underline{r}, \overline{r}\,]$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \leq \widehat{c}$). Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 \left[\, r - \underline{r} \,\right] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B optimally does not buy from $S1$ at any $p_1 \in (\,\underline{r}, \overline{r}\,]$ when $r = \underline{r}$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 < \underline{r}$, if B buys from $S1$ at this price, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \leq \widehat{c}$). Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] > 0$. Because $\phi_0(p_1) \leq \phi$ for all $p_1 < \underline{r}$, if B does not buy from $S1$, $S2$ will set $p_2 = \underline{r}$ (since $c_2 \leq \widehat{c}$). Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\, \underline{r} - \underline{r} \,] = 0$. Because $\pi_B^0(p_1, \underline{r}) < \pi_B^{n_1}(p_1, \underline{r})$, B optimally buys from $S1$ at any $p_1 < \underline{r}$ when $r = \underline{r}$.

2. We now prove that $S1$'s equilibrium strategy is optimal.

Because B always buys from $S1$ at price $p_1 = \underline{r}$, $S1$'s payoff in the putative equilibrium is:

$$\pi_1 (\, \underline{r} \,) \;=\; n_1 [\, \underline{r} - c_1] \;>\; 0\,.$$

Results Ci and Cii imply that this payoff exceeds the payoff that $S1$ earns by setting any price $p_1 > \underline{r}$. In particular, B does not buy from $S1$ if $p_1 > \underline{r}$, so $\pi_1 (p_1) = 0$ for all $p_1 > \underline{r}$.

Results Ci and Cii imply that B will buy from S_1 if $p_1 < \underline{r}$. Consequently, $S1$'s payoff from setting $p_1 < \underline{r}$ is:

$$\pi_1 (p_1) \;=\; n_1 [\, p_1 - c_1 \,] \;<\; n_1 [\, \underline{r} - c_1 \,] \;=\; \pi_1 (\, \underline{r} \,)\ .$$

Therefore, $S1$ maximizes her payoff by setting $p_1 = \underline{r}$.

D. Prove that $B2$'s beliefs satisfy Bayes Rule along the equilibrium path.

$$\Pr(r = \overline{r} \,|\, B \text{ buys at } p_1 = \underline{r}) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ buys at } p_1 = \underline{r})}{\Pr(\, B \text{ buys at } p_1 = \underline{r})}$$

$$= \frac{\Pr(\, B \text{ buys at } p_1 = \underline{r} \,|\, r = \overline{r}\,)\ \Pr(r = \overline{r}\,)}{\Pr(B \text{ buys at } p_1 = \underline{r} \,|\, r = \overline{r}\,)\ \Pr(r = \overline{r}) + \Pr(\, B \text{ buys at } p_1 = \underline{r} | \, r = \underline{r}\,)\ \Pr(r = \underline{r}\,)}$$

$$= \frac{1\,[\,\phi\,]}{1\,[\,\phi\,] + 1\,[\,1 - \phi\,]} \;=\; \phi \;=\; \phi_{n_1}(\,\underline{r}\,). \quad \blacksquare$$

Corollary 7. *Under the conditions specified in Theorem 7, B's equilibrium welfare is: (i) strictly higher under transparency than under privacy when* $r = \overline{r}$*; and (ii) the same under transparency and privacy when* $r = \underline{r}$*.*

Proof. Claim 3 implies that under privacy, B's equilibrium welfare is: (i) $n_2 [\, \underline{r} - \overline{r} \,]$ when $r = \overline{r}$; and (ii) 0 when $r = \underline{r}$.

Theorem 7 implies that under transparency, B's equilibrium welfare is: (i) 0 when $r = \underline{r}$; and (ii) $[\, n_1 + n_2 \,] [\, \underline{r} - \overline{r} \,]$ when $r = \overline{r}$. $\blacksquare$

Corollary 8. *Under the conditions specified in Theorem 7, transparency reduces the equilibrium payoff of $S1$ and leaves the equilibrium payoff of $S2$ unchanged.*

Proof. Claim 2 implies that under privacy, the payoffs of $S1$ and $S2$ are:

$$\pi_1^V \;=\; \phi\, n_1\, [\,\overline{r} - c_1\,] \quad \text{and} \quad \pi_2^V \;=\; n_2\, [\,\underline{r} - c_2\,]\,. \tag{24}$$

Theorem 7 shows that, under transparency, B will always buys from $S1$ and from $S2$ at price $\underline{r}$, so the equilibrium payoffs of $S1$ and $S2$ are:

$$\pi_1^T \;=\; n_1\, [\,\underline{r} - c_1] \;<\; \phi\, n_1\, [\,\overline{r} - c_1\,] \;=\; \pi_1^V \quad \text{and} \quad \pi_2^T = \; n_2\, [\,\underline{r} - c_2\,] \;=\; \pi_2^V\,. \tag{25}$$

The strict inequality in (25) holds because $c_1 > \widehat{c}$, by assumption. ■

Corollary 9. *Under the conditions specified in Theorem 7, transparency increases equilibrium industry welfare.*

Proof. Corollaries 4 and 5 and their proofs imply that under privacy, equilibrium expected industry welfare is:

$$W^V \;=\; \phi\, n_2\, [\,\overline{r} - \underline{r}\,] + \phi\, n_1\, [\,\overline{r} - c_1\,] + n_2\, [\,\underline{r} - c_2\,]\,. \tag{26}$$

(25) and the proof of Corollary 7 imply that under transparency, equilibrium industry welfare is:

$$W^T \;=\; \phi\, [\,n_1 + n_2\,]\, [\,\overline{r} - \underline{r}\,] + n_1\, [\,\underline{r} - c_1\,] + n_2\, [\,\underline{r} - c_2\,]\,. \tag{27}$$

(26) and (27) imply:

$$\begin{aligned}
W^T - W^V \;&=\; \phi\, n_1\, [\,\overline{r} - \underline{r}\,] + n_1\, [\,\underline{r} - c_1\,] - \phi\, n_1\, [\,\overline{r} - c_1\,] \\
&=\; n_1\, [\phi\, \overline{r} - \phi\, \underline{r} + \underline{r} - c_1 - \phi\, \overline{r} + \phi\, c_1] \\
&=\; n_1\, [\,-\,\phi\, \underline{r} + \underline{r} - c_1 + \phi\, c_1] \\
&=\; n_1\, [\,(1-\phi)\, \underline{r} - (1-\phi)\, c_1\,] \\
&=\; [\,1-\phi\,]\; n_1\, [\,\underline{r} - c_1\,] \;>\; 0\,. \quad ■
\end{aligned}$$

Theorem 8. *Under the conditions specified in Theorem 7, the equilibrium identified in the theorem is the unique pooling perfect Bayesian equilibrium.*

Proof. $S1$'s payoff is $n_1\, [\,\underline{r} - c_1] > 0$ in the identified pooling equilibrium. $S1$'s payoff is 0 in any pooling equilibrium in which B never buys from $S1$. $S1$'s payoff is also less than $n_1\, [\,\underline{r} - c_1]$ in any pooling equilibrium in which B always buys from $S1$ at $p_1 < \underline{r}$. Therefore, in any alternative candidate pooling equilibrium, B must always buy from $S1$ at $p_1 > \underline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 > \underline{r}$ is $n_1\, [\,\underline{r} - p_1\,] < 0$. B's welfare is non-negative if he does not buy from $S1$ at this price. Because B will not always buy from $S1$ at $p_1 > \underline{r}$, the equilibrium in which $S1$ sets $p_1 = \underline{r}$ is the unique pooling equilibrium under the specified conditions. ■

Setting 2A. $c_1 > \widehat{c}\,,\ c_2 > \widehat{c}$, and $n_1 > n_2$.

Theorem 9. *Suppose $n_1 > n_2$, $c_1 > c^* > \widehat{c}$, and $c_2 \geq \widehat{c}$. Then under transparency, a perfect Bayesian equilibrium exists in which: (i) S1 sets $\widehat{p}_1 \equiv \overline{r} - \frac{n_2}{n_1}[\,\overline{r} - \underline{r}\,]$; (ii) S2 sets $p_2 = \overline{r}$ if B buys from S1, whereas S2 sets $p_2 = \underline{r}$ if B does not buy from S1; (iii) B buys from S1 if $r = \overline{r}$, but does not buy from S1 if $r = \underline{r}$; and (iv) B buys from S2 if and only $p_2 \leq r$.*

Proof. The beliefs that support the identified equilibrium actions are: (i) $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$; (ii) $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$; (iii) $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$; and (iv) $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$.

The proof proceeds by backward induction. We first prove that $S2$'s equilibrium actions are optimal, given her beliefs. Then we prove that B's equilibrium actions are optimal, given $S2$'s beliefs. Next we prove that $S1$'s equilibrium actions are optimal. Finally, we verify that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

A. Prove that $S2$'s equilibrium actions are optimal.

First suppose B purchases n_1 units from $S1$ at price $\widehat{p}_1$. Because $\phi_{n_1}(\widehat{p}_1) = 1$, $S2$'s payoff is $\overline{r} - c_2 > 0$ if she sets $p_2 = \overline{r}$. Because $S2$'s secures payoff zero if she sets $p_2 > \overline{r}$ and secures payoff $p_2 - c_2 < \overline{r} - c_2$ if she sets $p_2 < \overline{r}$, $S2$ maximizes her payoff by setting $p_2 = \overline{r}$.

Now suppose B does not buy from $S1$ at price $\widehat{p}$. Because $\phi_0(\widehat{p}_1) = 0$, $S2$'s payoff is $\underline{r} - c_2 > 0$ if she sets $p_2 = \underline{r}$. Because $S2$ secures payoff zero if she sets $p_2 > \underline{r}$ and secures payoff $p_2 - c_2 < \underline{r} - c_2$ if she sets $p_2 < \underline{r}$, $S2$ maximizes her payoff by setting $p_2 = \underline{r}$.

B. Prove that B's equilibrium actions are optimal.

Because the game ends following B's interaction with $S2$, B maximizes his welfare by buying from $S2$ if and only if $p_2 \leq r$.

We now prove that B maximizes his welfare by buying from $S1$ at price $p_1 = \widehat{p}_1$ if $r = \overline{r}$ and not buying from $S1$ if $r = \underline{r}$.

First suppose $r = \overline{r}$. If B buys from $S1$ at price $\widehat{p}_1$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\widehat{p}_1) = 1$. Therefore, B's welfare is:

$$\begin{aligned} \pi_B^{n_1}(\widehat{p}_1, \overline{r}) &= n_1[\,\overline{r} - \widehat{p}_1\,] + n_2[\,\overline{r} - \overline{r}\,] \;=\; n_1[\,\overline{r} - \widehat{p}_1\,] \\ &= n_1\left[\overline{r} - \left(\overline{r} - \frac{n_2}{n_1}[\,\overline{r} - \underline{r}\,]\right)\right] \;=\; n_2[\,\overline{r} - \underline{r}\,]. \end{aligned} \tag{28}$$

If B does not buy from $S1$ at price $\widehat{p}_1$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\widehat{p}_1) = 0$. Therefore, B's welfare is:

$$\pi_B^0(\widehat{p}_1, \overline{r}) \;=\; n_2[\,\overline{r} - \underline{r}\,].$$

Because $\pi_B^{n_1}(\widehat{p}_1, \overline{r}) = \pi_B^0(\widehat{p}_1, \overline{r})$, B will buy from $S1$ at price $p_1 = \widehat{p}_1$ when $r = \overline{r}$.

Now suppose $r = \underline{r}$. If B buys from $S1$ at price $\widehat{p}_1$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\widehat{p}_1) = 1$. B will not buy from $S2$ at this price. Consequently, B's welfare is:

$$\pi_B^{n_1}(\widehat{p}_1, \underline{r}) \;=\; n_1 [\, \underline{r} - \widehat{p}_1 \,] \;=\; n_1 \left[\underline{r} - \left(\overline{r} - \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \right) \right]$$

$$=\; n_1 \, \underline{r} - n_1 \, \overline{r} + n_2 \, \overline{r} - n_2 \, \underline{r} \;=\; [\, n_2 - n_1 \,] \, [\, \overline{r} - \underline{r} \,] \;<\; 0 \, .$$

If B does not buy from $S1$ at price $\widehat{p}_1$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\widehat{p}_1) = 0$. Therefore, B's welfare is:

$$\pi_B^0(\widehat{p}_1, \underline{r}) \;=\; n_2 [\, \underline{r} - \underline{r} \,] \;=\; 0.$$

Because $\pi_B^0(\widehat{p}_1, \underline{r}) > \pi_B^{n_1}(\widehat{p}_1, \underline{r})$, B optimally does not buy from $S1$ at price $\widehat{p}_1$ when $r = \underline{r}$.
□

C. Prove that $S1$'s equilibrium action is optimal.

1. We begin by characterizing B's optimal response to out-of-equilibrium prices by $S1$.

Result C1. When $r = \overline{r}$, B optimally buys from $S1$ if $p_1 < \widehat{p}_1$ and does not buy from $S1$ if $p_1 > \widehat{p}_1$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \overline{r} \,] \;=\; n_1 [\, \overline{r} - p_1 \,] \;<\; 0 \, .$$

Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will either set $p_2 = \overline{r}$ (if $c_2 > \widehat{c}$) or $p_2 = \underline{r}$ (if $c_2 = \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;\geq\; \min \{ n_2 \, [\, \overline{r} - \overline{r} \,] \, , \, n_2 [\, \overline{r} - \underline{r} \,] \} \;=\; n_2 [\, \overline{r} - \overline{r} \,] \;=\; 0 \, .$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B will not buy from $S1$ at any price $p_1 > \overline{r}$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\widehat{p}_1, \overline{r}\,]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \overline{r} \,] \;=\; n_1 [\, \overline{r} - p_1 \,] \;<\; n_1 [\, \overline{r} - \widehat{p}_1 \,] \;=\; n_2 [\, \overline{r} - \underline{r} \,] \, ,$$

where the last equality follows from the analysis that underlies (28). Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \in (\widehat{p}_1, \overline{r}\,]$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \, .$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B optimally does not buy from $S1$ at price $p_1 \in (\widehat{p}_1, \overline{r}\,]$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1)$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is (again using the analysis that underlies (28)):

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \overline{r} \,] \;>\; n_1 [\, \overline{r} - \widehat{p}_1 \,] \;=\; n_2 [\, \overline{r} - \underline{r} \,] \, .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \in (\underline{r}, \widehat{p}_1)$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \, .$$

Because $\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r})$, B will buy from $S1$ for any $p_1 \in (\widehat{p}_1, \underline{r})$ when $r = \overline{r}$.

Now suppose that $S1$ sets $p_1 \leq \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ (because $c_2 \geq \widehat{c}$). Therefore, if B buys from $S1$, his welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \overline{r} \,] \;=\; n_1 [\, \overline{r} - p_1 \,] \;\geq\; n_1 [\, \overline{r} - \underline{r} \,] \, .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \leq \underline{r}$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \, .$$

$\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r})$ because $n_1 > n_2$. Consequently, B optimally buys from $S1$ at any price $p_1 \leq \underline{r}$ when $r = \overline{r}$. □

Result C2. When $r = \underline{r}$, B optimally: (i) does not buy from $S1$ if $p_1 > \underline{r}$ ($p_1 \neq \widehat{p}_1$); and (ii) buys from $S1$ if $p_1 \leq \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ when she sets $p_2 = \overline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\underline{r} - p_1] < 0$. Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ when she sets $p_1 > \overline{r}$, $S2$ will set $p_2 = \overline{r}$ (because $c_2 \geq \widehat{c}$). B will not buy from $S2$ at this price, so B's welfare is $\pi_B^0(p_1, \underline{r}) = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B optimally does not buy from $S1$ at any price $p_1 > \overline{r}$ when $r = \underline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \overline{r}\,]$ ($p_1 \neq \widehat{p}_1$). Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ when she sets price $p_2 = \overline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] < 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ when she sets $p_1 \in (\underline{r}, \overline{r}]$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\, \underline{r} - \underline{r} \,] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B will not buy from $S1$ if she sets $p_1 \in (\underline{r}, \overline{r}\,]$ ($p_1 \neq \widehat{p}_1$) when $r = \underline{r}$.

Now suppose that $S1$ sets $p_1 \leq \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ when she sets $p_2 = \overline{r}$ (because $c_2 \geq \widehat{c}$). Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] \geq 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \leq \underline{r}$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\, \underline{r} - \underline{r} \,] = 0$. Because $\pi_B^{n_1}(p_1, \underline{r}) \geq \pi_B^0(p_1, \underline{r})$, B optimally buys from $S1$ at price $p_1 \leq \underline{r}$ when $r = \underline{r}$. □

2. We now prove that $S1$'s equilibrium action is optimal.

When $S1$ sets $p_1 = \widehat{p}_1$, B buys from $S1$ if and only if $r = \overline{r}$. Consequently, $S1$'s payoff in the putative equilibrium is:

$$\pi_1(\widehat{p}_1) \;=\; \phi \, n_1 [\, \widehat{p}_1 - c_1 \,] \, .$$

Results C1 and C2 imply that B will not buy from $S1$ at price $p_1 > \widehat{p}_1$. Consequently, $S1$'s payoff is $\pi_1(p_1) = 0$ for all $p_1 > \widehat{p}_1$.

Results C1 and C2 imply that if $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1)$, B will buy from $S1$ if $r = \overline{r}$ and not buy from $S1$ if $r = \underline{r}$. Consequently, $S1$'s payoff is:

$$\pi_1(p_1) \;=\; \phi n_1 [\, p_1 - c_1 \,] \;<\; \phi\, n_1 [\, \widehat{p}_1 - c_1 \,] \,.$$

Results C1 and C2 imply that B will buy from $S1$ if $p_1 \le \underline{r}$. Consequently, $S1$'s payoff from setting $p_1 \le \underline{r}$ is:

$$\pi_1(p_1) \;=\; n_1 [\, p_1 - c_1 \,] \;\le\; n_1 [\, \underline{r} - c_1 \,] \;<\; \phi n_1 [\, \widehat{p}_1 - c_1 \,] \,.$$

The strict inequality here holds because:

$$\underline{r} - c_1 \;<\; \phi\, [\, \widehat{p}_1 - c_1 \,] \quad\Leftrightarrow\quad [\, 1 - \phi \,]\, c_1 \;>\; \underline{r} - \phi\, \widehat{p}_1 \;=\; \underline{r} - \phi \left[\overline{r} - \frac{n_2}{n_1} (\, \overline{r} - \underline{r} \,) \right]$$

$$\Leftrightarrow \quad [\, 1 - \phi \,]\, c_1 \;>\; \underline{r} - \phi\, \overline{r} + \phi\, \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \;=\; [\, 1 - \phi \,]\, \overline{r} - (\, \overline{r} - \underline{r} \,) + \phi\, \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,]$$

$$\Leftrightarrow \quad c_1 \;>\; \overline{r} - \frac{\overline{r} - \underline{r}}{1 - \phi} + \left[\frac{\phi}{1 - \phi} \right] \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \;=\; \widehat{c} + \phi\, \frac{n_2}{n_1} \left[\frac{\overline{r} - \underline{r}}{1 - \phi} \right] \;\equiv\; c^* \,.$$

Therefore, $S1$ maximizes her expected payoff by setting $p_1 = \widehat{p}_1$.

D. Prove that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

$$\Pr(r = \overline{r} \,|\, B \text{ buys at } \widehat{p}_1) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ buys at } \widehat{p}_1)}{\Pr(\, B \text{ buys at } \widehat{p}_1)}$$

$$= \; \frac{\Pr(B \text{ buys at } \widehat{p}_1 \,|\, r = \overline{r}\,)\, \Pr(r = \overline{r})}{\Pr(B \text{ buys at } \widehat{p}_1 \,|\, r = \overline{r}\,)\, \Pr(r = \overline{r}) + \Pr(\, B \text{ buys at } \widehat{p}_1 \,|\, r = \underline{r}\,)\, \Pr(r = \underline{r}\,)}$$

$$= \; \frac{1\, [\, \phi \,]}{1\, [\, \phi \,] + 0\, [\, 1 - \phi \,]} \;=\; 1 \;=\; \phi_{n_1}(\, \widehat{p}_1).$$

$$\Pr(r = \overline{r} \,|\, B \text{ does not buy at } \widehat{p}_1) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ does not buy at } \widehat{p}_1)}{\Pr(\, B \text{ does not buy at } \widehat{p}_1)}$$

$$= \; \frac{\Pr(B \text{ does not buy at } \widehat{p}_1 \,|\, r = \overline{r}\,)\, \Pr(r = \overline{r})}{\Pr(B \text{ does not buy at } \widehat{p}_1 \,|\, r = \overline{r}\,)\, \Pr(r = \overline{r}) + \Pr(\, B \text{ does not buy at } \widehat{p}_1 \,|\, r = \underline{r}\,)\, \Pr(r = \underline{r}\,)}$$

$$= \; \frac{0\, [\, \phi \,]}{0\, [\, \phi \,] + 1\, [\, 1 - \phi \,]} \;=\; 0 \;=\; \phi_0(\, \widehat{p}_1). \quad \blacksquare$$

Corollary 10. *Under the conditions specified in Theorem 9, B's equilibrium welfare is: (i) strictly higher under transparency than under privacy when $r = \overline{r}$; and (ii) the same under transparency and privacy when $r = \underline{r}$.*

Proof. Claim 3 demonstrates that under privacy, B's equilibrium welfare is 0 in the present setting both when $r = \overline{r}$ and when $r = \underline{r}$.

Under transparency, B's equilibrium welfare when $r = \overline{r}$ is:

$$n_1 [\, \overline{r} - \widehat{p}_1 \,] \;=\; n_1 \frac{n_2}{n_1} [\, \overline{r} \; - \underline{r} \,] \;=\; n_2 [\, \overline{r} \; - \underline{r} \,] \;>\; 0 \,. \tag{29}$$

Under transparency when $r = \underline{r}$, B's equilibrium welfare is $n_2 [\, \underline{r} - \underline{r} \,] = 0$. ■

Corollary 11. *Under the conditions specified in Theorem 9, transparency: (i) reduces the equilibrium payoff of* $S1$; *(ii) increases the equilibrium payoff of* $S2$*; and (iii) reduces the aggregate equilibrium payoff of the sellers if* $c_2 > \widehat{c}$*, and (iv) does not change the aggregate equilibrium payoff of the sellers if* $c_2 = \widehat{c}$*.*

Proof. Claim 2 implies that under privacy, the payoffs of $S1$ and $S2$ are:

$$\pi_1^V \;=\; \phi\, n_1 [\, \overline{r} - c_1 \,] \quad \text{and} \quad \pi_2^V \;=\; \phi\, n_2 [\, \overline{r} - c_2 \,] \,. \tag{30}$$

Under transparency, B buys from $S1$ at price $\widehat{p}_1$ (and so $S2$ will set $p_2 = \overline{r}$) if and only if $r = \overline{r}$. If $r = \underline{r}$, B does not buy from $S1$ at price $\widehat{p}_1$ and so $S2$ sets $p_2 = \underline{r}$. Therefore, $S1$'s equilibrium payoff is lower under transparency:

$$\pi_1^T \;=\; \phi\, n_1 [\, \widehat{p}_1 - \, c_1 \,] \;<\; \phi\, n_1 [\, \overline{r} \, - c_1 \,] \;=\; \pi_1^V \,. \tag{31}$$

$S2$'s equilibrium payoff is higher under transparency:

$$\pi_2^T \;=\; \phi\, n_2 [\, \overline{r} \, - c_2 \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2] \;>\; \phi\, n_2 [\, \overline{r} \, - c_2 \,] \;=\; \pi_2^V \,. \tag{32}$$

(30) implies that the aggregate equilibrium payoff of the sellers under privacy is:

$$\pi_S^V \;=\; \phi\, n_1 [\, \overline{r} - c_1 \,] + \phi\, n_2 [\, \overline{r} - c_2 \,] \,. \tag{33}$$

(31) and (32) imply that under transparency, the aggregate equilibrium payoff of the sellers is:

$$\pi_S^T \;=\; \phi\, n_1 [\, \widehat{p}_1 - \, c_1 \,] + \phi\, n_2 [\, \overline{r} \, - c_2 \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2 \,] \,. \tag{34}$$

(33) and (34) imply:

$$\begin{aligned}
\pi_S^T - \pi_S^V \;=\;& \phi\, n_1 [\, \widehat{p}_1 - \, c_1 \,] + \phi\, n_2 [\, \overline{r} \, - c_2 \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2 \,] \\
& - (\, \phi\, n_1 [\, \overline{r} - c_1 \,] + \phi\, n_2 [\, \overline{r} - c_2 \,] \,) \\
=\;& \phi\, n_1 [\, \widehat{p}_1 - \, c_1 \,] - \phi\, n_1 [\, \overline{r} - c_1 \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2 \,] \\
=\;& \phi\, n_1 \, \widehat{p}_1 - \, \phi\, n_1 \, c_1 - \phi\, n_1 \, \overline{r} + \phi\, n_1 \, c_1 \, + [\, 1 - \phi \,]\, n_2 [\underline{r} \, - \, c_2 \,] \\
=\;& \phi\, n_1 \, \widehat{p}_1 - \phi\, n_1 \, \overline{r} \, + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2 \,] \\
=\;& \phi\, n_1 [\, \widehat{p}_1 - \overline{r} \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, c_2] \\
\leq\;& \phi\, n_1 [\, \widehat{p}_1 - \overline{r} \,] + [\, 1 - \phi \,]\, n_2 [\, \underline{r} \, - \, \widehat{c}] \\
=\;& \phi\, n_1 \left[\overline{r} - \frac{n_2}{n_1} (\, \overline{r} \, - \underline{r} \,) - \overline{r} \right] + [\, 1 - \phi \,]\, n_2 \left[\underline{r} \, - \left(\overline{r} \, - \frac{\overline{r} \, - \underline{r}}{1 - \phi} \right) \right] \\
=\;& - \phi\, n_2 [\, \overline{r} - \, \underline{r} \,] - [\, 1 - \phi \,]\, n_2 [\, \overline{r} - \, \underline{r} \,] + n_2 [\, \overline{r} - \, \underline{r} \,] \;=\; 0 \,.
\end{aligned}$$

The weak inequality here holds strictly when $c_2 > \widehat{c}$, and holds as an equality when $c_2 = \widehat{c}$. ■

Corollary 12. *Under the conditions specified in Theorem 9, transparency increases equilibrium industry welfare.*

Proof. Under privacy, $S1$ and $S2$ both charge $\overline{r}$ and B's welfare is 0. Therefore, industry welfare is aggregate supplier welfare:

$$W^V \;=\; \phi\, n_1 \,[\, \overline{r} - c_1\,] + \phi\, n_2\, [\, \overline{r} - c_2\,]\,. \tag{35}$$

Under transparency, B's welfare is as specified in (29) with probability ϕ, and the sellers receive the payoffs in (31) and (32). Therefore, under transparency, equilibrium industry welfare is:

$$W^T \;=\; \phi\, n_2\, [\, \overline{r} - \underline{r}\,] + \phi\, n_1\, [\, \widehat{p}_1 - c_1\,] + \phi\, n_2\, [\, \overline{r} - c_2\,] + [\, 1 - \phi\,]\, n_2\, [\, \underline{r} - c_2\,]\,. \tag{36}$$

(35) and (36) imply:

$$\begin{aligned}
W^T - W^V \;&=\; \phi\, n_2\, [\, \overline{r} - \underline{r}\,] + \phi\, n_1\, [\, \widehat{p}_1 - c_1\,] + \phi\, n_2\, [\, \overline{r} - c_2\,] + [\, 1 - \phi\,]\, n_2\, [\, \underline{r} - c_2\,] \\
&\quad - (\, \phi\, n_1\, [\, \overline{r} - c_1\,] + \phi\, n_2\, [\, \overline{r} - c_2\,]\,) \\
&=\; \phi\, n_2\, [\, \overline{r} - \underline{r}\,] + \phi\, n_1\, [\, \widehat{p}_1 - c_1\,] + [\, 1 - \phi\,]\, n_2\, [\, \underline{r} - c_2\,] - \phi\, n_1\, [\, \overline{r} - c_1\,] \\
&=\; \phi\, n_2\, [\, \overline{r} - \underline{r}\,] + \phi\, n_1 \left[\, \overline{r} - \frac{n_2}{n_1}\, (\, \overline{r} - \underline{r}\,) - c_1 \right] + [\, 1 - \phi\,]\, n_2\, [\, \underline{r} - c_2\,] - \phi\, n_1\, [\, \overline{r} - c_1\,] \\
&=\; [\, 1 - \phi\,]\, n_2\, [\, \underline{r} - c_2\,] \;>\; 0\,. \quad \blacksquare
\end{aligned}$$

Theorem 10. *Suppose $n_1 > n_2$, $c^* > c_1 > \widehat{c}$, and $c_2 \geq \widehat{c}$. Then a separating perfect Bayesian equilibrium does not exist under transparency.*

Proof. Initially suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \overline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 1$ and $\phi_0(\widetilde{p}_1) = 0$. Consequently, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1\, [\, \overline{r} - \widetilde{p}_1\,] + n_2\, [\, \overline{r} - \overline{r}\,] \;=\; n_1\, [\, \overline{r} - \widetilde{p}_1\,]\,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{0}(\widetilde{p}_1, \overline{r}) \;=\; n_2\, [\, \overline{r} - \underline{r}\,]\,.$$

Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1\, [\, \overline{r} - \widetilde{p}_1\,] \;\geq\; n_2\, [\, \overline{r} - \underline{r}\,] \quad \Leftrightarrow \quad \widetilde{p}_1 \;\leq\; \overline{r} - \frac{n_2}{n_1}\, [\, \overline{r} - \underline{r}\,]\,. \tag{37}$$

B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \underline{r}$ (and subsequently does not buy from $S2$ when she sets $p_2 = \overline{r}$) is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \;=\; n_1 [\, \underline{r} - \widetilde{p}_1 \,] \,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{0}(\widetilde{p}_1, \underline{r}) \;=\; n_2 [\, \underline{r} - \underline{r} \,] \;=\; 0 \,.$$

Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\, \underline{r} - \widetilde{p}_1 \,] \;<\; 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \;>\; \underline{r} \,. \tag{38}$$

These conditions together imply that $\widetilde{p}_1 \in (\, \underline{r} \,, \overline{r} - \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,]]$ in any candidate separating equilibrium.

We now demonstrate that $\pi_1 (\, \underline{r} \,) > \pi_1 (\widetilde{p}_1)$, i.e., that $S1$ secures a higher payoff if she sets $p_1 = \underline{r}$ than if she sets $p_1 = \widetilde{p}_1 \in (\, \underline{r} \,, \overline{r} - \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,]]$. To do so, we first show that B will always buy from $S1$ if she sets price $p_1 = \underline{r}$. Because $\phi_{n_1}(\, \underline{r} \,) = \phi$ and $c_2 \geq \widehat{c}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ when she sets $p_1 = \underline{r}$. Because $\phi_0(\, \underline{r} \,) = 0$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ when she sets $p_1 = \underline{r}$. Therefore, when $r = \overline{r}$, B's welfare if he buys from $S1$ at this price is:

$$\pi_B^{n_1}(\, \underline{r}, \overline{r} \,) \;=\; n_1 [\, \overline{r} - \underline{r} \,] + n_2 [\, \overline{r} \;- \overline{r} \,] \;=\; n_1 [\, \overline{r} - \underline{r} \,] \,.$$

B's corresponding welfare if he does not buy from $S1$ at this price is:

$$\pi_B^{0}(\, \underline{r}, \overline{r} \,) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;<\; n_1 [\, \overline{r} - \underline{r} \,] \,.$$

Because $\pi_B^{n_1}(\, \underline{r}, \overline{r} \,) > \pi_B^{0}(\, \underline{r}, \overline{r} \,)$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 = \underline{r}$ is:

$$\pi_B^{n_1}(\, \underline{r}, \underline{r} \,) \;=\; n_1 [\, \underline{r} \;- \underline{r} \,] \;=\; 0 \,.$$

B's corresponding welfare if he does not buy from $S1$ at this price is:

$$\pi_B^{0}(\, \underline{r}, \underline{r} \,) \;=\; n_2 [\, \underline{r} \;- \underline{r} \,] \;=\; 0 \,.$$

Because $\pi_B^{n_1}(\, \underline{r}, \underline{r} \,) = \pi_B^{0}(\, \underline{r}, \underline{r} \,)$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $c = \underline{r}$.

It remains to demonstrate that $\pi_1 (\, \underline{r} \,) > \pi_1 (\widetilde{p}_1)$. To do so, it is sufficient to prove that:

$$\pi_1 (\, \underline{r} \,) \;=\; n_1 [\, \underline{r} - c_1] \;>\; \phi \, n_1 [\, \widehat{p}_1 \;- c_1 \,] \;=\; \pi_1 (\widetilde{p}_1) \tag{39}$$

or equivalently:

$$[\, 1 - \phi \,] \, c_1 \;<\; \underline{r} - \phi \, \widehat{p}_1 . \tag{40}$$

From (37):

$$\underline{r} - \phi \, \widehat{p}_1 \;\geq\; \underline{r} - \phi \left[\, \overline{r} - \frac{n_2}{n_1} \, (\, \overline{r} - \underline{r} \,) \right] .$$

Therefore, (40) holds if:

$$[\, 1 - \phi \,] \, c_1 \;<\; \underline{r} - \phi \, \left[\, \overline{r} - \frac{n_2}{n_1} \, (\, \overline{r} - \underline{r} \,) \right]$$

$$= \underline{r} - \phi \overline{r} + \phi \frac{n_2}{n_1} [\overline{r} - \underline{r}] = [1-\phi]\overline{r} - (\overline{r} - \underline{r}) + \phi \frac{n_2}{n_1} [\overline{r} - \underline{r}]$$

$$\Leftrightarrow \quad c_1 < \overline{r} - \frac{\overline{r} - \underline{r}}{1-\phi} + \phi \frac{n_2}{n_1} [\overline{r} - \underline{r}] = \widehat{c} + \phi \frac{n_2}{n_1} \left[\frac{\overline{r} - \underline{r}}{1-\phi} \right] \equiv c^* . \tag{41}$$

(41) holds because $c_1 < c^*$ by hypothesis.

Now suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \underline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 0$ and $\phi_0(\widetilde{p}_1) = 1$. Consequently, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \overline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) = n_1 [\overline{r} - \widetilde{p}_1] + n_2 [\overline{r} - \underline{r}] .$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \overline{r}) = n_2 [\overline{r} - \overline{r}] = 0 .$$

Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\overline{r} - \widetilde{p}_1] + n_2 [\overline{r} - \underline{r}] < 0 \quad \Leftrightarrow \quad \widetilde{p}_1 > \overline{r} + \frac{n_2}{n_1} [\overline{r} - \underline{r}] > \overline{r} . \tag{42}$$

B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \underline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) = n_1 [\underline{r} - \widetilde{p}_1] + n_2 [\underline{r} - \underline{r}] = n_1 [\underline{r} - \widetilde{p}_1] .$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ when she sets $p_2 = \overline{r}$) is 0. Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\underline{r} - \widetilde{p}_1] > 0 \quad \Leftrightarrow \quad \widetilde{p}_1 < \underline{r} . \tag{43}$$

(43) provides a contradiction of (42). ■

Theorem 11. *Suppose $c_1 > \widehat{c}$ and $c_2 \geq \widehat{c}$. Then a pooling perfect Bayesian equilibrium does not exist under transparency.*

<u>Proof</u>. Because $c_2 \geq \widehat{c}$ and $S2$'s beliefs must satisfy Bayes Rule along the equilibrium path, $S2$ will always set $p_2 = \overline{r}$ in any pooling equilibrium. B will buy from $S2$ at this price if and only if $r = \overline{r}$, and so secures zero welfare from his interaction with $S2$.

First suppose B always buys from $S1$. Then the putative equilibrium price cannot exceed $\underline{r}$. Otherwise, B could increase his welfare by not buying from $S1$ when $r = \underline{r}$. Therefore, $S1$'s payoff in the putative equilibrium cannot exceed $\pi_1(\underline{r}) = n_1 [\underline{r} - c_1]$. However, because $c_1 > \widehat{c}$, we have a contradiction: By setting $p_1 = \overline{r}$, $S1$ can increase her payoff to $\phi \, n_1 [\overline{r} - c_1] > n_1 [\underline{r} - c_1]$.[1]

[1]This inequality holds because $c_1 > \widehat{c}$, by assumption.

Now suppose B never buys from $S1$. Then the price $S1$ sets must exceed $\overline{r}$. Otherwise, B would buy from $S1$ when $r = \overline{r}$. Therefore, $S1$'s payoff in the putative equilibrium is zero. But because $c_1 > \widehat{c}$, we have a contradiction: By setting $p_1 = \overline{r}$, $S1$ can increase her payoff to $\phi\, n_1 [\,\overline{r} - c_1\,] > 0$. ■

Setting 2B. $c_1 > \widehat{c}\,,\ c_2 > \widehat{c}$, and $n_1 \le n_2$.

Theorem 1 indicates that a separating perfect Bayesian equilibrium does not exist under transparency in this setting when $n_2 \ge n_1$.

Theorem 11 indicates that a pooling perfect Bayesian equilibrium does not exist under transparency in this setting.

Observation. Theorems 1, 9, 10, and 11 imply that a perfect Bayesian equilibrium does not exist in this setting if $n_2 \ge n_1$ or if $n_1 > n_2$ and $c_1 \in (\,\widehat{c},\, c^*\,)$. The unique perfect Bayesian equilibrium when $c_1 > c^* > \widehat{c}$ and $n_1 > n_2$ is a separating equilibrium in which $S1$ sets $\widehat{p}_1 \equiv \overline{r} - \frac{n_2}{n_1}[\,\overline{r} - \underline{r}\,]$.

Setting 3. $c_1 \le \widehat{c}$ and $c_2 \le \widehat{c}$.

Theorem 12. *Suppose $c_1 \le \widehat{c}$ and $c_2 \le \widehat{c}$. Then under transparency, a pooling perfect Bayesian equilibrium exists in which: (i) $S1$ sets $\underline{r}$; (ii) $S2$ sets $\underline{r}$; and (iii) B always buys from both $S1$ and $S2$ at these prices.*

Proof. The following beliefs support the identified equilibrium actions: (i) $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$; (ii) $\phi_{n_1}(p_1) = \phi$ for all $p_1 \le \underline{r}$; (iii) $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$; and (iv) $\phi_0(p_1) = 0$ for all $p_1 < \overline{r}$.

The proof proceeds by backward induction. We first prove that $S2$'s equilibrium action is optimal, given his beliefs. Then we prove that B's equilibrium actions are optimal, given $S2$'s beliefs. Next we prove that $S1$'s equilibrium action is optimal. Finally, we verify that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

A. Prove that $S2$'s equilibrium action is optimal.

$\phi_{n_1}(\,\underline{r}\,) = \phi$. Therefore, after B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will maximize her expected payoff by acting as she does under privacy. Claim 1 implies that because $c_2 \le \widehat{c}$, $S2$ will set $p_2 = \underline{r}$.

$\phi_0(\,\underline{r}\,) = 0$. Therefore, after B does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ secures payoff: (i) 0 if she sets $p_2 > \underline{r}$; and (ii) $p_2 - c_2$ if she sets $p_2 \le \underline{r}$. Consequently, $S2$ will set $p_2 = \underline{r}$.

B. Prove that B's equilibrium actions are optimal.

Because the game ends following B's interaction with $S2$, B maximizes his welfare by buying from from $S2$ at price $p_2 = \underline{r}$.

We now prove that B maximizes his payoff by buying from $S1$ at price $p_1 = \underline{r}$.

First suppose $r = \overline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 < \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{n_1}(p_1 = \underline{r}, \overline{r}) \;=\; n_1 [\overline{r} - \underline{r}] + n_2 [\overline{r} - \underline{r}] \;=\; [n_1 + n_2][\overline{r} - \underline{r}].$$

If B does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\underline{r}) = 0$. Therefore, B's welfare is:

$$\pi_B^0(p_1 = \underline{r}, \overline{r}) \;=\; n_2 [\overline{r} - \underline{r}] \;<\; [n_1 + n_2][\overline{r} - \underline{r}].$$

Because $\pi_B^{n_1}(p_1 = \underline{r}, \overline{r}) > \pi_B^0(p_1 = \underline{r}, \overline{r})$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

Now suppose $r = \underline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 < \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{n_1}(p_1 = \underline{r}, \underline{r}) \;=\; n_1 [\underline{r} - \underline{r}] + n_2 [\underline{r} - \underline{r}] \;=\; 0.$$

If B does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\underline{r}) = 0$. Therefore, B's welfare is:

$$\pi_B^0(p_1 = \underline{r}, \underline{r}) \;=\; n_2 [\underline{r} - \underline{r}] \;=\; 0.$$

Because $\pi_B^0(p_1 = \underline{r}, \underline{r}) = \pi_B^{n_1}(p_1 = \underline{r}, \underline{r})$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \underline{r}$. $\square$

C. Prove that $S1$'s equilibrium action is optimal.

1. We begin by characterizing B's optimal response to out-of-equilibrium prices by $S1$.

Result C1. When $r = \overline{r}$, B optimally does not buy from $S1$ at any price $p_1 > \underline{r}$ and buys from $S1$ at any price $p_1 < \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\overline{r} - p_1] + n_2 [\overline{r} - \overline{r}] \;=\; n_1 [\overline{r} - p_1] \;<\; 0.$$

Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 \le \widehat{c}$, by assumption). Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\overline{r} - \underline{r}] \;>\; 0.$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B will not buy from $S1$ at price $p_1 > \overline{r}$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \overline{r}]$ and $n_2 \ge n_1$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\overline{r} - p_1] + n_2 [\overline{r} - \overline{r}] \;=\; n_1 [\overline{r} - p_1] \;<\; n_1 [\overline{r} - \underline{r}].$$

Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}]$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 \in (\underline{r}, \overline{r}]$. Consequently, B's expected payoff is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;\geq\; n_1 [\, \overline{r} - \underline{r} \,] \, .$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B will not buy from $S1$ at price $p_1 \in (\underline{r}, \overline{r}\,]$ when $r = \overline{r}$ and $n_2 \geq n_1$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \overline{r}\,]$ and $n_1 > n_2$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \overline{r} \,] \;=\; n_1 [\, \overline{r} - p_1 \,] \, .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}\,]$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 \in (\underline{r}, \overline{r}\,]$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;>\; n_1 [\, \overline{r} - p_1 \,] \quad \Leftrightarrow \quad p_1 \;>\; \overline{r} - \frac{n_2}{n_1} [\, \overline{r} - \overline{r} \,] \;>\; \underline{r} \, . \tag{44}$$

Inequality (44) holds because $p_1 > \underline{r}$ by assumption in this case. Therefore, $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, so B will not buy from $S1$ at price $p_1 \in (\underline{r}, \overline{r}\,]$ when $r = \overline{r}$ and $n_1 > n_2$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ (because $c_2 < \widehat{c}$). Therefore, if B buys from $S1$ at price $p_1 < \underline{r}$, his welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \;=\; n_1 [\, \overline{r} - p_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \;>\; [\, n_1 + n_2 \,] [\, \overline{r} - \underline{r} \,] \, .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \underline{r}$, if B does not buy from $S1$ at price $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \;<\; [\, n_1 + n_2 \,] [\, \overline{r} - \underline{r} \,] \, .$$

Because $\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r})$, B will buy from $S1$ at price $p_1 > \underline{r}$ when $r = \overline{r}$. $\square$

Result C2. When $r = \underline{r}$, B optimally does not buy from $S1$ at any $p_1 > \underline{r}$ and buys from $S1$ at any $p_1 < \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ at price $p_2 = \overline{r}$. Consequently, his welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] < 0$. Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 < \widehat{c}$). Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\, \underline{r} - \underline{r} \,] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B will not buy from $S1$ at any $p_1 > \overline{r}$ when $r = \underline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \overline{r}\,]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ when she sets $p_2 = \overline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] < 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r}\,]$, if B does not buy from $S1$ at price $p_1 \in (\underline{r}, \overline{r}\,]$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\, \underline{r} - \underline{r} \,] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B will not buy from $S1$ at any $p_1 \in (\underline{r}, \overline{r}\,]$ when $r = \underline{r}$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$ (because $c_2 < \widehat{c}$) if B buys from $S1$ at price $p_1 < \underline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\, \underline{r} - p_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] > 0$. Because $\phi_0(p_1) = 0$ for all $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 < \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\underline{r} - \underline{r}] = 0$. Because $\pi_B^{n_1}(p_1, \underline{r}) > \pi_B^0(p_1, \underline{r})$, B will buy from $S1$ at any price $p_1 < \underline{r}$ when $r = \underline{r}$. □

2. We now prove that $S1$'s equilibrium action is optimal.

Because B always buys from $S1$ at $p_1 = \underline{r}$, $S1$'s payoff is:

$$\pi_1(\underline{r}) \;=\; n_1 [\underline{r} - c_1] \;>\; 0\,.$$

Results C1 and C2 imply that B will not buy from $S1$ at any price $p_1 > \underline{r}$. Consequently, $S1$'s payoff is $\pi_1(p_1) = 0$ for all $p_1 > \underline{r}$.

Results C1 and C2 imply that B will buy from $S1$ at any price $p_1 \leq \underline{r}$. Consequently, $S1$'s payoff when she sets $p_1 < \underline{r}$ is:

$$\pi_1(p_1) \;=\; n_1 [p_1 - c_1] \;<\; n_1 [\underline{r} - c_1]\,.$$

Therefore, $S1$ maximizes her payoff by setting $p_1 = \underline{r}$.

D. <u>Prove that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path</u>.

$$\Pr(r = \overline{r} \mid B \text{ buys at } p_1 = \underline{r}) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ buys at } p_1 = \underline{r})}{\Pr(B \text{ buys at } p_1 = \underline{r})}$$

$$= \frac{\Pr(B \text{ buys at } p_1 = \underline{r} \mid r = \overline{r})\, \Pr(r = \overline{r})}{\Pr(B \text{ buys at } p_1 = \underline{r} \mid r = \overline{r})\, \Pr(r = \overline{r}) + \Pr(B \text{ buys at } p_1 = \underline{r} \mid r = \underline{r})\, \Pr(r = \underline{r})}$$

$$= \frac{1\,[\phi]}{1\,[\phi] + 1\,[1-\phi]} \;=\; \phi \;=\; \phi_{n_1}(\underline{r}). \quad \blacksquare$$

Corollary 13. *Under the conditions specified in Theorem 12, B's equilibrium welfare is the same under transparency and under privacy.*

<u>Proof</u>. Claim 1 implies that under privacy, $S1$ and $S2$ both charge $\underline{r}$. Both sellers also charge $\underline{r}$ under transparency. Furthermore, B always buys from $S1$ and from $S2$ in both settings. Therefore, B's equilibrium welfare in both settings is: (i) $[n_1 + n_2][\overline{r} - \underline{r}] > 0$ when $r = \overline{r}$; and (ii) 0 when $r = \underline{r}$. ■

Corollary 14. *Under the conditions specified in Theorem 12, S1 and S2 each secure the same equilibrium payoff under transparency that it secures under privacy.*

<u>Proof</u>. Claim 1 implies that under privacy, $S1$ and $S2$ both charge $\underline{r}$. Both sellers also charge $\underline{r}$ under transparency. Furthermore, B always buys from $S1$ and from $S2$ in both settings. Therefore, Si's equilibrium payoff in both regimes is $n_i [\underline{r} - c_i] > 0$ for $i = 1, 2$. ■

Theorem 13. *Under the conditions specified in Theorem 12, the equilibrium identified in the theorem is the unique pooling perfect Bayesian equilibrium.*

Proof. $S1$'s payoff is $n_1 [\underline{r} - c_1] > 0$ in the identified pooling equilibrium. $S1$'s payoff is 0 in any pooling equilibrium in which B never buys from $S1$ at any given price p_1. $S1$'s payoff is less than $n_1 [\underline{r} - c_1]$ in any pooling equilibrium in which B always buys from $S1$ at a price $p_1 < \underline{r}$. Therefore, in any alternative candidate pooling equilibrium, B must always buy from $S1$ at price $p_1 > \underline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 > \underline{r}$ is $n_1 [\underline{r} - p_1] < 0$. B's welfare is non-negative if he does not buy from $S1$ at this price. Because B will not always buy from $S1$ at a price $p_1 > \underline{r}$, the equilibrium in which $S1$ sets $p_1 = \underline{r}$ is the unique pooling equilibrium under the specified conditions. ■

Theorem 14. *Suppose $c_1 < \widehat{c}$ and $c_2 < \widehat{c}$. Then a separating perfect Bayesian equilibrium does not exist under transparency.*

Proof. Initially suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \overline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 1$ and $\phi_0(\widetilde{p}_1) = 0$. Consequently, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$. $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1 [\overline{r} - \widetilde{p}_1] + n_2 [\overline{r} - \overline{r}] \;=\; n_1 [\overline{r} - \widetilde{p}_1].$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \overline{r}) \;=\; n_2 [\overline{r} - \underline{r}].$$

Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\overline{r} - \widetilde{p}_1] \;\geq\; n_2 [\overline{r} - \underline{r}] \;\;\Leftrightarrow\;\; \widetilde{p}_1 \;\leq\; \overline{r} - \frac{n_2}{n_1} [\overline{r} - \underline{r}]. \tag{45}$$

B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) when $r = \underline{r}$ is $\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) = n_1 [\underline{r} - \widetilde{p}_1]$. B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is $\pi_B^0(\widetilde{p}_1, \underline{r}) = n_2 [\underline{r} - \underline{r}] = 0$. Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\underline{r} - \widetilde{p}_1] \;<\; 0 \;\;\Leftrightarrow\;\; \widetilde{p}_1 \;>\; \underline{r}. \tag{46}$$

This completes this part of the proof if $n_2 \geq n_1$ because (46) contradicts (45) in this case.

Now suppose $n_1 > n_2$. We will demonstrate that $S1$ secures a higher payoff by setting $p_1 = \underline{r}$ than by setting $\widetilde{p}_1 \in (\underline{r}, \overline{r} - \frac{n_2}{n_1} [\overline{r} - \underline{r}]]$ when $n_1 > n_2$. We first show that B will always buy from $S1$ when she sets $p_1 = \underline{r}$. Because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 < \widehat{c}$, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at price $p_1 = \underline{r}$. Because $\phi_0(\underline{r}) = 0$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 = \underline{r}$. Therefore, when $r = \overline{r}$, B's welfare if he buys from $S1$ is:

$$\pi_B^{n_1}(\underline{r}, \overline{r}) \;=\; n_1 [\overline{r} - \underline{r}] + n_2 [\overline{r} - \underline{r}] \;=\; [n_1 + n_2][\overline{r} - \underline{r}].$$

B's corresponding payoff if he does not buy from $S1$ is:

$$\pi_B^0(\underline{r}, \overline{r}) \;=\; n_2 [\, \overline{r} - \underline{r} \,] \,.$$

Because $\pi_B^{n_1}(\underline{r}, \overline{r}) > \pi_B^0(\underline{r}, \overline{r})$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 = \underline{r}$ is:

$$\pi_B^{n_1}(\underline{r}, \underline{r}) \;=\; n_1 [\, \underline{r} - \underline{r} \,] + n_2 [\, \underline{r} - \underline{r} \,] \;=\; 0.$$

B's corresponding payoff if he does not buy from $S1$ is:

$$\pi_B^0(\underline{r}, \underline{r}) \;=\; n_2 [\, \underline{r} - \underline{r} \,] \;=\; 0 \,.$$

Because $\pi_B^{n_1}(\underline{r}, \underline{r}) \geq \pi_B^0(\underline{r}, \underline{r})$, B buys from $S1$ at $p_1 = \underline{r}$ when $r = \underline{r}$.

We now establish that $S1$ will not set $p_1 = \widetilde{p}_1$ under the specified conditions because $S1$ can earn a strictly higher payoff by setting $p_1 = \underline{r}$. Because B always buys from $S1$ when she sets $p_1 = \underline{r}$ under the specified conditions, $S1$'s payoff when she sets $p_1 = \underline{r}$ is:

$$\pi_1 (\underline{r}) \;=\; n_1 [\, \underline{r} - c_1 \,] \;>\; \phi \, n_1 [\, \overline{r} - c_1 \,] \;>\; \phi \, n_1 [\, \widetilde{p}_1 - c_1 \,] \;=\; \pi_1 (\widetilde{p}_1) \,.$$

The first inequality here reflects Claim 1 and the hypothesis that $c_1 < \widehat{c}$. The second inequality reflects (45), which implies:

$$\widetilde{p}_1 \;\leq\; \overline{r} - \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \;<\; \overline{r} \,.$$

Now suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \underline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1) = 0$ and $\phi_0(\widetilde{p}_1) = 1$. Consequently, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \overline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

First suppose $r = \overline{r}$. B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1 [\, \overline{r} - \widetilde{p}_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^0(\widetilde{p}_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \overline{r} \,] \;=\; 0 \,.$$

Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if**:**

$$n_1 [\, \overline{r} - \widetilde{p}_1 \,] + n_2 [\, \overline{r} - \underline{r} \,] \;\geq\; 0 \;\;\Leftrightarrow\;\; \widetilde{p}_1 \;\leq\; \overline{r} + \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \,.$$

Now suppose $r = \underline{r}$. B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \;=\; n_1 [\, \underline{r} - \widetilde{p}_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] \;=\; n_1 [\, \underline{r} - \widetilde{p}_1 \,] \,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ when she sets $p_2 = \overline{r}$) is 0. Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ if and only if:

$$n_1 [\, \underline{r} - \widetilde{p}_1] \;\geq\; 0 \;\;\Leftrightarrow\;\; \widetilde{p}_1 \;\leq\; \underline{r} \,. \qquad (47)$$

(47) implies that in the postulated separating equilibrium, $S1$'s price cannot exceed $\underline{r}$ if B is to always buy from $S1$. However, B will always buy from $S1$ when $p_1 \leq \underline{r}$ under the

specified conditions. Therefore, the postulated separating equilibrium cannot exist. ■

Setting 4A. $c_1 \leq \widehat{c}$, $c_2 > \widehat{c}$, and $n_1 \geq n_2$.

Theorem 15. *Suppose $n_1 \geq n_2$, $c_1 \leq \widehat{c}$, and $c_2 > \widehat{c}$. Then under transparency, a pooling perfect Bayesian equilibrium exists in which: (i) $S1$ sets $p_1 = \underline{r}$; (ii) $S2$ sets $p_2 = \overline{r}$; (iii) B always buys from $S1$; and (iv) B buys from $S2$ if and only if $r = \overline{r}$.*

Proof. The beliefs that support the identified equilibrium actions are: (i) $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$; (ii) $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$; (iii) $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$; and (iv) $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$.

The proof proceeds by backward induction. We first prove that $S2$'s equilibrium action is optimal, given her beliefs. Then we prove that B's equilibrium actions are optimal, given $S2$'s beliefs. Next we prove that $S1$'s equilibrium action is optimal. Finally, we verify that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

A. Prove that $S2$'s equilibrium action is optimal.

$\phi_{n_1}(\underline{r}) = \phi$. Therefore, because $c_2 > \widehat{c}$, $S2$ will set $p_2 = \overline{r}$ after B buys from $S1$ at price $p_1 = \underline{r}$.

B. Prove that B's equilibrium actions are optimal.

Because the game ends following B's interaction with $S2$, B maximizes his welfare by buying from $S2$ at price $p_2 = \overline{r}$ if and only if $r = \overline{r}$.

We now prove that B maximizes his welfare by always buying from $S1$ at price $p_1 = \underline{r}$.

First suppose $r = \overline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 > \widehat{c}$. Therefore, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \ = \ n_1 [\, \overline{r} - \underline{r}\,] + n_2 [\, \overline{r} - \overline{r}\,] \ = \ n_1 [\, \overline{r} - \underline{r}\,] .$$

If B does not buy from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\underline{r}) = 0$. Therefore, B's welfare is:

$$\pi_B^{0}(p_1, \overline{r}) \ = \ n_2 [\, \overline{r} - \underline{r}\,] \ \leq \ n_1 [\, \overline{r} - \underline{r}\,] .$$

Because $\pi_B^{n_1}(p_1, \overline{r}) \geq \pi_B^{0}(p_1, \overline{r})$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

Now suppose $r = \underline{r}$. If B buys from $S1$ at price $p_1 = \underline{r}$, $S2$ will set $p_2 = \overline{r}$ because $\phi_{n_1}(\underline{r}) = \phi$ and $c_2 > \widehat{c}$. B will not buy from $S2$ at this price. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \underline{r}) \ = \ [\, \underline{r} - \underline{r}\,] \ = \ 0 .$$

If B does not buy from $S1$ when she sets $p_1 = \underline{r}$, $S2$ will set $p_2 = \underline{r}$ because $\phi_0(\underline{r}) = 0$. Therefore, B's welfare is:

$$\pi_B^{0}(p_1, \underline{r}) \ = \ n_2 [\, \underline{r} - \underline{r}\,] \ = \ 0 .$$

Because $\pi_B^{0}(p_1, \underline{r}) = \pi_B^{n_1}(p_1, \underline{r})$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \underline{r}$. □

C. Prove that $S1$'s equilibrium action is optimal.

1. We begin by characterizing B's optimal response to out-of-equilibrium prices by $S1$.

Result C1. When $r = \overline{r}$, B optimally does not buy from $S1$ if $p_1 > \widehat{p}_1 \equiv \overline{r} - \frac{n_2}{n_1}[\overline{r} - \underline{r}]$ and buys from $S1$ if $p_1 \leq \widehat{p}_1$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \ = \ n_1[\overline{r} - p_1] + n_2[\overline{r} - \overline{r}] \ = \ n_1[\overline{r} - p_1] \ < \ 0 .$$

Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \overline{r}$ (because $c_2 > \widehat{c}$). Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \ = n_2[\overline{r} - \overline{r}] \ = \ 0 .$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B will not buy from $S1$ at price $p_1 > \overline{r}$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\widehat{p}_1, \overline{r}]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \ = \ n_1[\overline{r} - p_1] + n_2[\overline{r} - \overline{r}] \ = \ n_1[\overline{r} - p_1] \ < \ n_1[\overline{r} - \widehat{p}_1] \ = \ n_2[\overline{r} - \underline{r}] .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \in (\widehat{p}_1, \overline{r}]$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \ = \ n_2[\overline{r} - \underline{r}] .$$

Because $\pi_B^0(p_1, \overline{r}) > \pi_B^{n_1}(p_1, \overline{r})$, B will not buy from $S1$ at any $p_1 \in (\widehat{p}_1, \overline{r}]$ when $r = \overline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1]$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$. Consequently, B's welfare is:

$$\pi_B^{n_1}(p_1, \overline{r}) \ = \ n_1[\overline{r} - p_1] + n_2[\overline{r} - \overline{r}] \ \geq \ n_1[\overline{r} - \widehat{p}_1] \ = \ n_2[\overline{r} - \underline{r}] .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 \in (\underline{r}, \widehat{p}_1]$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \ = \ n_2[\overline{r} - \underline{r}] .$$

Because $\pi_B^{n_1}(p_1, \overline{r}) \geq \pi_B^0(p_1, \overline{r})$, B will buy from $S1$ at any $p_1 \in (\overline{r}, \widehat{p}_1]$ when $r = \overline{r}$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$, if B buys from $S1$, $S2$ will set $p_2 = \overline{r}$ (because $c_2 > \widehat{c}$). Therefore, if B buys from $S1$, his payoff is:

$$\pi_B^{n_1}(p_1, \overline{r}) \ = \ n_1[\overline{r} - p_1] + n_2[\overline{r} - \overline{r}] \ > \ n_1[\overline{r} - \underline{r}] .$$

Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$ at price $p_1 < \underline{r}$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is:

$$\pi_B^0(p_1, \overline{r}) \ = \ n_2[\overline{r} - \underline{r}] \ < \ n_1[\underline{r} - \overline{r}] .$$

Because $\pi_B^{n_1}(p_1, \overline{r}) > \pi_B^0(p_1, \overline{r})$, B will buy from $S1$ at price $p_1 < \underline{r}$ when $r = \overline{r}$. $\square$

Result C2. When $r = \underline{r}$, B optimally: (i) buys from $S1$ whenever $p_1 < \underline{r}$; and (ii) does not buy from $S1$ whenever $p_1 > \underline{r}$.

Proof. Initially suppose that $S1$ sets $p_1 > \overline{r}$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ at price $p_2 = \overline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\underline{r} - p_1] < 0$. Because $\phi_0(p_1) = \phi$ for all $p_1 > \overline{r}$, if B does not buy from $S1$ at price $p_1 > \overline{r}$, $S2$ will set $p_2 = \overline{r}$ (because $c_2 > \widehat{c}$). B will not buy from $S2$ at this price, so B's welfare is $\pi_B^0(p_1, \underline{r}) = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B will not buy from $S1$ at price $p_1 > \overline{r}$ when $r = \underline{r}$.

Next suppose that $S1$ sets $p_1 \in (\underline{r}, \overline{r})$. Because $\phi_{n_1}(p_1) = 1$ for all $p_1 > \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ at price $p_2 = \overline{r}$. Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\underline{r} - p_1] < 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \in (\underline{r}, \overline{r})$, if B does not buy from $S1$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\underline{r} - \underline{r}] = 0$. Because $\pi_B^0(p_1, \underline{r}) > \pi_B^{n_1}(p_1, \underline{r})$, B will not buy from $S1$ when she sets any price $p_1 \in (\underline{r}, \overline{r})$ when $r = \underline{r}$.

Now suppose that $S1$ sets $p_1 < \underline{r}$. Because $\phi_{n_1}(p_1) = \phi$ for all $p_1 \leq \underline{r}$, if B buys from $S1$ at this price, he will subsequently not buy from $S2$ at price $p_2 = \overline{r}$. (Recall $c_2 > \widehat{c}$). Consequently, B's welfare is $\pi_B^{n_1}(p_1, \underline{r}) = n_1 [\underline{r} - p_1] > 0$. Because $\phi_0(p_1) = 0$ for all $p_1 \leq \overline{r}$, if B does not buy from $S1$, $S2$ will set $p_2 = \underline{r}$. Consequently, B's welfare is $\pi_B^0(p_1, \underline{r}) = n_2 [\underline{r} - \underline{r}] = 0$. Because $\pi_B^{n_1}(p_1, \underline{r}) > \pi_B^0(p_1, \underline{r})$, B will buy from $S1$ at price $p_1 > \underline{r}$ when $r = \underline{r}$. □

2. We now prove that $S1$'s equilibrium action is optimal.

When $S1$ sets $p_1 = \underline{r}$, B always buys from $S1$. Consequently, $S1$'s payoff is:

$$\pi_1(\underline{r}) \;=\; n_1 [\underline{r} - c_1] \;>\; 0\,.$$

Results C1 and C2 imply that B will not buy from $S1$ at price $p_1 > \widehat{p}_1$. Consequently, $S1$'s payoff is $\pi_1(p_1) = 0$ for all $p_1 > \widehat{p}_1$.

Results C1 and C2 imply that if $S1$ sets $p_1 \in (\underline{r}, \widehat{p}_1]$, B will buy from $S1$ if $r = \overline{r}$ and not buy from $S1$ if $r = \underline{r}$. Consequently, $S1$'s payoff from setting $p_1 \in (\underline{r}, \widehat{p}_1]$ is:

$$\pi_1(p_1) \;=\; \phi\, n_1 [p_1 - c_1] \;<\; \phi\, n_1 [\overline{r} - c_1] \;\leq\; n_1 [\underline{r} - c_1]\,.$$

The strict inequality here holds because $p_1 < \overline{r}$. The weak inequality follows from Claim 1 and the maintained assumption that $c_1 \leq \widehat{c}$.

Results C1 and C2 imply that $S1$'s payoff from setting $p_1 < \underline{r}$ is:

$$\pi_1(p_1) \;=\; n_1 [p_1 - c_1] \;<\; n_1 [\underline{r} - c_1]\,.$$

Therefore, $S1$ maximizes her payoff by setting $p_1 = \underline{r}$.

D. Prove that $S2$'s beliefs satisfy Bayes Rule along the equilibrium path.

$$\Pr(r = \overline{r} \,|\, B \text{ buys at } \underline{r}) \;=\; \frac{\Pr(r = \overline{r} \text{ and } B \text{ buys at } \underline{r})}{\Pr(B \text{ buys at } \underline{r})}$$

$$= \frac{\Pr(\,B \text{ buys at } \underline{r} \,|\, r = \overline{r}\,) \; \Pr(r = \overline{r}\,)}{\Pr(\,B \text{ buys at } \underline{r} \,|\, r = \overline{r}\,) \; \Pr(r = \overline{r}\,) + \Pr(B \text{ buys at } \underline{r} \,|\, r = \underline{r}\,) \; \Pr(r = \underline{r}\,)}$$

$$= \frac{1\,[\,\phi\,]}{1\,[\,\phi\,] + 1\,[\,1 - \phi\,]} \;=\; \phi \;=\; \phi_{n_1}(\,\underline{r}\,). \quad \blacksquare$$

Corollary 15. *Under the conditions specified in Theorem 15, transparency does not alter B's equilibrium welfare or the equilibrium payoffs of $S1$ or $S2$.*

Proof. From Claim 1 and Theorem 15, equilibrium behavior is identical in the two regimes: $S1$ sets $p_1 = \underline{r}$; $S2$ sets $p_2 = \overline{r}$; B always buys from $S1$ at price $p_1 = \underline{r}$; and B buys from $S2$ at price $p_1 = \overline{r}$ if and only if $r = \overline{r}$. ■

Theorem 16. *Under the conditions specified in Theorem 15, the equilibrium identified in the theorem is the unique pooling perfect Bayesian equilibrium.*

Proof. $S1$'s payoff is $n_1\,[\,\underline{r} - c_1\,] > 0$ in the identified pooling equilibrium. $S1$'s payoff is 0 in any pooling equilibrium in which B never buys from $S1$. $S1$'s payoff is also less than $n_1\,[\,\underline{r} - c_1]$ in any pooling equilibrium in which B always buys from $S1$ at price $p_1 < \underline{r}$. Therefore, in any alternative candidate pooling equilibrium, B must always buy from $S1$ at a price $p_1 > \underline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 > \underline{r}$ is $n_1\,[\,\underline{r} - p_1\,] < 0$. B's welfare is non-negative if he does not buy from $S1$ at this price. Because B will not always buy from $S1$ when she sets $p_1 > \underline{r}$, the equilibrium in which $S1$ sets $p_1 = \underline{r}$ is the unique pooling equilibrium under the specified conditions. ■

Theorem 17. *Suppose $c_1 \leq \widehat{c}$ and $c_2 > \widehat{c}$. Then a separating perfect Bayesian equilibrium does not exist under transparency.*

Proof. Initially suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \overline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\,\widetilde{p}_1\,) = 1$ and $\phi_0(\,\widetilde{p}_1\,) = 0$. Consequently, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ when she sets price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \leq r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\,\widetilde{p}_1, \overline{r}\,) \;=\; n_1\,[\,\overline{r} - \widetilde{p}_1\,] + n_2\,[\,\overline{r} - \overline{r}\,] \;=\; n_1\,[\,\overline{r} - \widetilde{p}_1\,]\,. \tag{48}$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{0}(\,\widetilde{p}_1, \overline{r}\,) \;=\; n_2\,[\,\overline{r} - \underline{r}\,]\,. \tag{49}$$

(48) and (49) imply that B will buy from $S1$ at $\widetilde{p}_1$ if and only if:

$$n_1\,[\,\overline{r}-\widetilde{p}_1\,] \;\geq\; n_2\,[\,\overline{r}-\underline{r}\,] \quad\Leftrightarrow\quad \widetilde{p}_1 \;\leq\; \overline{r}-\frac{n_2}{n_1}\,[\,\overline{r}-\underline{r}\,]\,. \tag{50}$$

B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) when $r = \underline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \;=\; n_1\,[\,\underline{r}-\widetilde{p}_1\,]\,. \tag{51}$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ is:

$$\pi_B^{0}(\widetilde{p}_1, \underline{r}) \;=\; n_2\,[\,\underline{r}-\underline{r}\,] \;=\; 0\,. \tag{52}$$

(51) and (52) imply that B will not buy from $S1$ at $\widetilde{p}_1$ if and only if:

$$n_1\,[\,\underline{r}-\widetilde{p}_1\,] \;<\; 0 \quad\Leftrightarrow\quad \widetilde{p}_1 \;>\; \underline{r}\,. \tag{53}$$

(53) provides a contradiction of (50) when $n_2 \geq n_1$.

We now demonstrate that $S1$ earns a higher payoff by setting $p_1 = \underline{r}$ than by setting $\widetilde{p}_1 \in (\,\underline{r}, \overline{r} - \frac{n_2}{n_1}\,[\,\overline{r}-\underline{r}\,]]$ when $n_1 > n_2$ (and hence, $\widetilde{p}_1 < \overline{r}$). We first show that B will always buy from $S1$ at price $p_1 = \underline{r}$. Because $\phi_{n_1}(\,\underline{r}\,) = \phi$ and $c_2 \geq \widehat{c}$, $S2$ will set $p_2 = \overline{r}$ if B buys from $S1$ at price $p_1 = \underline{r}$. Because $\phi_0(\,\underline{r}\,) = 0$, $S2$ will set $p_2 = \underline{r}$ if B does not buy from $S1$ at price $p_1 = \underline{r}$. Therefore, when $r = \overline{r}$, B's welfare if he buys from $S1$ at this price is:

$$\pi_B^{n_1}(\,\underline{r}, \overline{r}\,) \;=\; n_1\,[\,\overline{r}\,-\,\underline{r}\,] + n_2\,[\,\overline{r}\,-\,\underline{r}\,] \;=\; [\,n_1+n_2\,]\,[\,\overline{r}\,-\,\underline{r}\,]\,.$$

B's corresponding welfare if he does not buy from $S1$ at this price is:

$$\pi_B^{0}(\,\underline{r}, \overline{r}\,) \;=\; n_2\,[\,\overline{r}\,-\,\underline{r}\,]\,.$$

Because $\pi_B^{n_1}(\,\underline{r}, \overline{r}\,) > \pi_B^{0}(\,\underline{r}, \overline{r}\,)$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \overline{r}$.

When $r = \underline{r}$, B's welfare if he buys from $S1$ at price $p_1 = \underline{r}$ (and subsequently does not buy from $S2$ at price $p_2 = \overline{r}$) is:

$$\pi_B^{n_1}(\underline{r}, \underline{r}) \;=\; n_1\,[\,\underline{r}\,-\underline{r}\,] \;=\; 0\,.$$

B's corresponding welfare if he does not buy from $S1$ at this price is:

$$\pi_B^{0}(\underline{r}, \underline{r}) \;=\; n_2\,[\,\underline{r}\,-\underline{r}\,] \;=\; 0\,.$$

Because $\pi_B^{n_1}(\,\underline{r}, \underline{r}\,) \geq \pi_B^{0}(\,\underline{r}, \underline{r}\,)$, B will buy from $S1$ at price $p_1 = \underline{r}$ when $r = \underline{r}$.

We now demonstrate that $S1$ secures a higher payoff by setting $p_1 = \underline{r}$ than by setting $p_1 = \widetilde{p}_1$. Because B always buys from $S1$ when she sets $p_1 = \underline{r}\widetilde{p}_1$, $S1$ secures payoff $\pi_1\,(\,\underline{r}\,) = n_1\,[\,\underline{r}\,-c_1]$ if she sets $p_1 = \underline{r}$. If $S1$ sets $p_1 = \widetilde{p}_1$, she secures payoff:

$$\pi_1\,(\widetilde{p}_1\,) \;=\; \phi\, n_1\,[\,\widetilde{p}_1\,-c_1\,] \;<\; \phi\, n_1\,[\,\overline{r}\,-c_1\,] \;\leq\; n_1\,[\,\underline{r}\,-c_1\,] \;=\; \pi_1\,(\,\underline{r}\,)\,.$$

The strict inequality here holds because $\widetilde{p}_1 < \overline{r}$. The weak inequality reflects Claim 1 and the maintained assumption that $c_1 \leq \widehat{c}$.

Now suppose a separating equilibrium exists in which B buys from $S1$ at price $\widetilde{p}_1$ if and only if $r = \underline{r}$. Because beliefs must satisfy Bayes Rule along the equilibrium path, $\phi_{n_1}(\widetilde{p}_1\,) = 0$ and $\phi_0(\widetilde{p}_1\,) = 1$. Consequently, $S2$ will set $p_2 = \underline{r}$ if B buys from $S1$ at price $\widetilde{p}_1$, whereas $S2$ will set $p_2 = \overline{r}$ if B does not buy from $S1$ at price $\widetilde{p}_1$.

B will buy from $S2$ if and only if $p_2 \le r$. Therefore, B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \overline{r}) \;=\; n_1 [\, \overline{r} \,-\, \widetilde{p}_1] + n_2 [\, \overline{r} - \underline{r} \,] \,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ is:

$$\pi_B^{0}(\widetilde{p}_1, \overline{r}) \;=\; n_2 [\, \overline{r} - \overline{r} \,] = \; 0.$$

Therefore, B will buy from $S1$ at price $\widetilde{p}_1$ when $r = \overline{r}$ if and only if:

$$n_1 [\, \overline{r} \,-\, \widetilde{p}_1] + n_2 [\, \overline{r} - \underline{r} \,] \;\ge\; 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \;\le\; \overline{r} \,-\, \frac{n_2}{n_1} [\, \overline{r} - \underline{r} \,] \,. \tag{54}$$

B's welfare if he buys from $S1$ at price $\widetilde{p}_1$ when $r = \underline{r}$ is:

$$\pi_B^{n_1}(\widetilde{p}_1, \underline{r}) \;=\; n_1 [\, \underline{r} \,-\, \widetilde{p}_1 \,] + n_2 [\, \underline{r} - \underline{r} \,] \;=\; n_1 [\, \underline{r} \,-\, \widetilde{p}_1 \,] \,.$$

B's welfare if he does not buy from $S1$ at price $\widetilde{p}_1$ (and subsequently buys from $S2$ at price $p_2 = \underline{r}$) when $r = \underline{r}$ is 0. Therefore, B will not buy from $S1$ at price $\widetilde{p}_1$ when $r = \underline{r}$ if and only if:

$$n_1 [\, \underline{r} \,-\, \widetilde{p}_1 \,] \;<\; 0 \quad \Leftrightarrow \quad \widetilde{p}_1 \;>\, \underline{r} \,. \tag{55}$$

(55) implies that in the postulated separating equilibrium, B will not buy from $S1$ at any price above $\underline{r}$ when $r = \underline{r}$. (54) implies that when $r = \overline{r}$, B will buy from $S1$ when she sets a price above $\underline{r}$. Therefore, the postulated separating equilibrium cannot exist. ■

Setting 4B. $c_1 \le \widehat{c}$, $c_2 > \widehat{c}$, and $n_1 < n_2$.

Observe that Theorem 17 holds both in Setting 4A and in Setting 4B.

Theorem 18. *Suppose $c_1 \le \widehat{c}$, $c_2 > \widehat{c}$, and $n_1 < n_2$. Further suppose a pooling perfect Bayesian equilibrium exists under transparency.*[2] *Then this equilibrium is the one specified in Theorem 15.*

Proof. The proof is identical to the proof of Theorem 16. ■

[2] It can be shown that, under the conditions specified in Theorem 18, a pooling perfect Bayesian equilibrium exists (does not exist) if out-of-equilibrium beliefs are passive (if $\phi_0(\underline{r}) = 0$).